HARMONIC FORM & DOUBLE ANGLE FORMULAS

A2 Unit 3: Pure Mathematics B

WJEC past paper questions: 2010 – 2017

Total marks available 93 (approximately 1 hour 50 minutes)

1.

(a) Find all values of θ in the range $0° \leqslant \theta \leqslant 360°$ satisfying

$$2\cos 2\theta = 9\cos\theta + 7.$$ [5]

(b) (i) Express $5\sin x - 12\cos x$ in the form $R\sin(x-\alpha)$, where R and α are constants with $R > 0$ and $0° < \alpha < 90°$.

(ii) Use your results to part (i) to find the least value of

$$\frac{1}{5\sin x - 12\cos x + 20}.$$

Write down a value for x for which this least value occurs. [6]

(Summer 10)

a) $2\cos 2\theta = 9\cos\theta + 7$

Use $\cos 2\theta = 2\cos^2\theta - 1$

$2(2\cos^2\theta - 1) = 9\cos\theta + 7$

$4\cos^2\theta - 9\cos\theta - 9 = 0$

xac	+b
−36	−9

−12, 3

$4\cos^2\theta - 12\cos\theta + 3\cos\theta - 9 = 0$

$4\cos\theta(\cos\theta - 3) + 3(\cos\theta - 3) = 0$

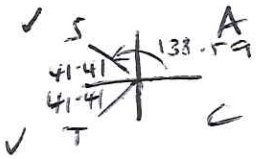
$(\cos\theta - 3)(4\cos\theta + 3) = 0$

$\cos\theta = 3$ ∄ no solutions

$\cos\theta = -3/4$

$\theta = 138.590°$

$\theta = 221.41°$



b) i) $5\sin x - 12\cos x = R\sin(x-\alpha)$

$5\sin x - 12\cos x = R\sin x\cos\alpha - R\cos x\sin\alpha$

$5 = R\cos\alpha$ (1) $\quad 12 = R\sin\alpha$ (2)

(1)² + (2)² $\quad R^2(\sin^2\alpha + \cos^2\alpha) = 5^2 + 12^2$

$\sin^2\alpha + \cos^2\alpha = 1$

$R^2 = 5^2 + 12^2$, $R^2 = 169$ $\quad R = 13$

(2)/(1) $\quad \frac{R\sin\alpha}{R\cos\alpha} = \frac{12}{5}$, $\tan\alpha = \frac{12}{5}$ $\quad \alpha = 67.38°$

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$13\sin(x - 67{\cdot}38°)$

ii) Require least value of $\dfrac{1}{5\sin x - 12\cos x + 20}$

$\therefore$ require least value of $\dfrac{1}{13\sin(x-67{\cdot}38) + 20}$

least value occurs when denominator is as big as possible $= \dfrac{1}{13+20} = \dfrac{1}{33}$

This occurs when $\sin(x - 67{\cdot}38) = 1$

$x - 67{\cdot}38 = 90°$

$x = 157{\cdot}38°$

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2.

(a) Find all values of x in the range $0° \leqslant x \leqslant 180°$ satisfying

$$\tan 2x = 4\tan x.$$ [5]

(b) Express $7\cos\theta + 24\sin\theta$ in the form $R\cos(\theta - \alpha)$, where R and α are constants with $R > 0$ and $0° < \alpha < 90°$.
Hence, find all values of θ in the range $0° \leqslant \theta \leqslant 360°$ satisfying

$$7\cos\theta + 24\sin\theta = 16.$$ [6]

(Summer 11)

a) $\tan 2x = 4\tan x$ $\quad 0° \leqslant x \leqslant 180°$

use $\tan 2x \equiv \dfrac{2\tan x}{1-\tan^2 x}$

$$\frac{2\tan x}{1-\tan^2 x} = 4\tan x$$

$$2\tan x = 4\tan x(1-\tan^2 x)$$

$$4\tan^3 x - 2\tan x = 0$$

$$2\tan x(2\tan^2 x - 1) = 0$$

$2\tan x = 0$ $\quad$ $2\tan^2 x - 1 = 0$

$\tan x = 0$ $\quad$ $\tan^2 x = \frac{1}{2}$

$x = 0°, x = 180°$ $\quad$ $\tan x = \pm\sqrt{\frac{1}{2}}$

$x = 35.26°, x = 144.74°$

b) $7\cos\theta + 24\sin\theta = R\cos(\theta - \alpha)$

$7\cos\theta + 24\sin\theta = R\cos\theta\cos\alpha + R\sin\theta\sin\alpha$

$7 = R\cos\alpha$ (1) $\quad 24 = R\sin\alpha$ (2)

(1)^2 + (2)^2 $\quad R^2(\sin^2\alpha + \cos^2\alpha) = 7^2 + 24^2$

$R^2 = 625$, $R = 25$

(2)/(1) $\quad \dfrac{R\sin\alpha}{R\cos\alpha} = \dfrac{24}{7}$

$\tan\alpha = \frac{24}{7}$ $\quad \alpha = 73.74°$

$= 25\cos(\theta - 73.74°)$

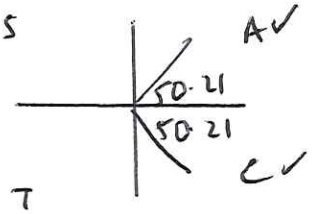
equation becomes $25\cos(\theta - 73.74) = 16$

$\cos(\theta - 73.74) = 16/25$

$\theta - 73.74 = 50.21°$

$\theta - 73.74 = 50.210, 309.79°$ (309.79° too big)

-50.21



$\theta = 50.21 + 73.74$

$\theta = -50.21 + 73.74$

$\theta = 23.53°, 123.95°$

3.

(a) Find all values of θ in the range $0° \leqslant \theta \leqslant 360°$ satisfying

$$8\cos 2\theta + 6 = \cos^2\theta + \cos\theta.$$ [6]

(b) Express $\sqrt{15}\cos\theta - \sin\theta$ in the form $R\cos(\theta + \alpha)$, where R and α are constants with $R > 0$ and $0° < \alpha < 90°$.
Hence find all values of θ in the range $0° \leqslant \theta \leqslant 360°$ satisfying

$$\sqrt{15}\cos\theta - \sin\theta = 3.$$ [6]

(Summer 13)

a) $8\cos 2\theta + 6 = \cos^2\theta + \cos\theta$

use $\cos 2\theta \equiv \cos^2\theta - \sin^2\theta$

$8(\cos^2\theta - \sin^2\theta) + 6 = \cos^2\theta + \cos\theta$

$7\cos^2\theta - 8\sin^2\theta - \cos\theta + 6 = 0$

use $\sin^2\theta = 1 - \cos^2\theta$

$7\cos^2\theta - 8(1 - \cos^2\theta) - \cos\theta + 6 = 0$

$15\cos^2\theta - \cos\theta - 2 = 0$

x ac	+b
-30	1

$-6, 5$

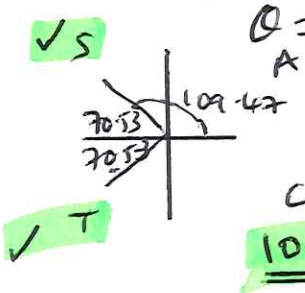
$15\cos^2\theta + 5\cos\theta - 6\cos\theta - 2 = 0$

$5\cos\theta(3\cos\theta + 1) - 2(3\cos\theta + 1) = 0$

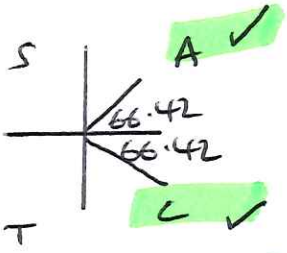
$(3\cos\theta + 1)(5\cos\theta - 2) = 0$

$\cos\theta = -\frac{1}{3}$ $\qquad$ $\cos\theta = \frac{2}{5}$

$\theta = 109.470°$ $\qquad$ $\theta = 66.42°$



$109.470°, 250.53°$



$66.42°, 293.58°$

b) $\sqrt{15}\cos\theta - \sin\theta = R\cos(\theta + \alpha)$

$\sqrt{15}\cos\theta - \sin\theta = R\cos\theta\cos\alpha - R\sin\theta\sin\alpha$

$\sqrt{15} = R\cos\alpha$ (1) $\qquad$ $1 = R\sin\alpha$ (2)

(1)^2 + (2)^2 $\qquad R^2(\sin^2\alpha + \cos^2\alpha) = (\sqrt{15})^2 + 1^2$

$\sin^2\alpha + \cos^2\alpha = 1$

$\therefore R^2 = 16$, $\quad R = 4$

(2)/(1) $\qquad \frac{R\sin\alpha}{R\cos\alpha} = \frac{1}{\sqrt{15}}$, $\quad \tan\alpha = \frac{1}{\sqrt{15}}$

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$\alpha = \tan^{-1}\left(\frac{1}{\sqrt{15}}\right)$

$\alpha = 14.48°$

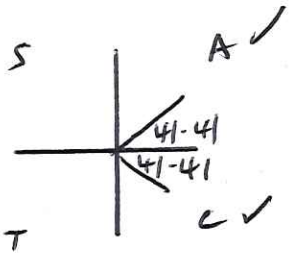
$4\cos(\theta + 14.48°)$

Require $4\cos(\theta + 14.48) = 3$

$\cos(\theta + 14.48) = 3/4$

$\theta + 14.48 = \cos^{-1}(3/4)$

$\theta + 14.48° = 41.41°$



$\theta + 14.48 = 41.41°$ $\quad$ $\theta + 14.48 = 318.59°$

$\theta = 26.93°$ $\quad$ $\theta = 304.11°$

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4.

(a) Find all values of θ in the range $0° \leqslant \theta \leqslant 360°$ satisfying

$$4\cos 2\theta = 1 - 2\sin\theta.$$ [6]

(b) (i) Express $8\sin x + 15\cos x$ in the form $R\sin(x + \alpha)$, where R and α are constants with $R > 0$ and $0° < \alpha < 90°$.

(ii) Find all values of x in the range $0° \leqslant x \leqslant 360°$ satisfying

$$8\sin x + 15\cos x = 11.$$

(iii) Find the greatest possible value for k so that

$$8\sin x + 15\cos x = k$$

has solutions. Give a reason for your answer. [7]

(Summer 12)

a) $4\cos 2\theta = 1 - 2\sin\theta$ use $\cos 2\theta \equiv 1 - 2\sin^2\theta$

$4(1 - 2\sin^2\theta) = 1 - 2\sin\theta$

$0 = 8\sin^2\theta - 2\sin\theta - 3$

$\times ac$	$+b$
-24	-2

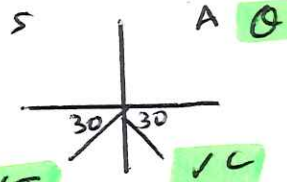
$-6, 4$

$0 = 8\sin^2\theta + 4\sin\theta - 6\sin\theta - 3$

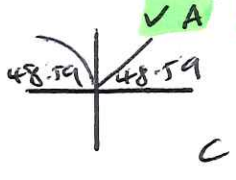
$0 = 4\sin\theta(2\sin\theta + 1) - 3(2\sin\theta + 1)$

$0 = (2\sin\theta + 1)(4\sin\theta - 3)$

$\sin\theta = -\frac{1}{2}$ $\quad$ $\sin\theta = \frac{3}{4}$



✓A $\theta = -30°$ ✓S ✓A $\theta = 48.59°$



✓T ✓C

$\theta = 48.59°, 134.41°, 210°, 330°$

b) i) $8\sin x + 15\cos x = R\sin(x + \alpha)$

$8\sin x + 15\cos x = R\sin x\cos\alpha + R\cos x\sin\alpha$

① $8 = R\cos\alpha$ ② $15 = R\sin\alpha$

① ^{2} + ② ^{2} $R^2(\sin^2\alpha + \cos^2\alpha) = 8^2 + 15^2$, $R^2 = 289$

$R = 17$

②/① $\frac{R\sin\alpha}{R\cos\alpha} = \frac{15}{8}$, $\tan\alpha = \frac{15}{8}$ $\quad$ $\alpha = 61.93°$

$17\sin(x + 61.93°)$

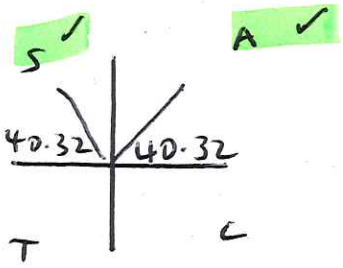
ii) $8\sin x + 15\cos x = 11$

becomes $17\sin(x + 61.93) = 11$

$\sin(x + 61.93) = \frac{11}{17}$

$x + 61.93 = \sin^{-1}\left(\frac{11}{17}\right)$

$= \underline{40.32^\circ}$



$x + 61.93^\circ = 40.32^\circ, 139.67^\circ, 400.32^\circ$

↓ TOO SMALL

$\underline{x = 77.75^\circ, 338.39^\circ}$

iii) Require greatest value of $8\sin x + 15\cos x = k$

$\Rightarrow$ greatest value of $17(\sin(x + 61.93)) = k$

$\underline{k = 17}$

because max value of $\sin(\text{---}) = 1$

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5. (a) Find all values of x in the range $0° \leqslant x \leqslant 180°$ satisfying

$$\tan 2x = 3\cot x. \qquad [4]$$

(b) (i) Express $21\sin\theta - 20\cos\theta$ in the form $R\sin(\theta - \alpha)$, where R and α are constants with $R > 0$ and $0° < \alpha < 90°$.

(ii) Use your results to part (i) to find the greatest value of

$$\frac{1}{21\sin\theta - 20\cos\theta + 31}.$$

Write down a value for θ for which this greatest value occurs. [6]

(Summer 14)

a) $\tan 2x = 3\cot x$ use $\tan 2x \equiv \frac{2\tan x}{1-\tan^2 x}$ & $\cot x \equiv \frac{1}{\tan x}$

$$\frac{2\tan x}{1-\tan^2 x} = \frac{3}{\tan x}$$

$$2\tan^2 x = 3 - 3\tan^2 x$$

$$5\tan^2 x = 3$$

$$\tan^2 x = \frac{3}{5}$$

$$\tan x = \pm\sqrt{3/5}$$

$x = 37.76°, 142.24°$

b i) $21\sin\theta - 20\cos\theta = R\sin(\theta - \alpha)$

$21\sin\theta - 20\cos\theta = R\sin\theta\cos\alpha - R\cos\theta\sin\alpha$

$21 = R\cos\alpha$ (1) $\quad 20 = R\sin\alpha$ (2)

(1)^2 + (2)^2 = $R^2(\sin^2\alpha + \cos^2\alpha) = 20^2 + 21^2$

$\sin^2\alpha + \cos^2\alpha = 1$

$\therefore R^2 = 841$

$R = \sqrt{841}$, $R = 29$

(2)/(1) gives $\frac{R\sin\alpha}{R\cos\alpha} = \frac{20}{21} \rightarrow \tan\alpha = \frac{20}{21}$

$\alpha = \tan^{-1}\left(\frac{20}{21}\right)$

$\alpha = 43.6°$

$29\sin(\theta - 43.6°)$

ii) greatest value of $\frac{1}{21\sin\theta - 20\cos\theta + 31}$

= greatest value of $\frac{1}{29\sin(\theta - 43.6) + 31}$

occurs when $\sin(\theta - 43.6) = -1$

$\theta = 270 + 43.6$

$\theta = 313.6°$

$= \frac{1}{2}$

6. (a) Find all values of x in the range $0° \leqslant x \leqslant 180°$ satisfying

$$\tan(x + 45°) = 8\tan x.$$ [5]

(b) (i) Express $\sqrt{13}\sin\theta - 6\cos\theta$ in the form $R\sin(\theta - \alpha)$, where R and α are constants with $R > 0$ and $0° < \alpha < 90°$.

(ii) Find all values of θ in the range $0° \leqslant \theta \leqslant 360°$ satisfying

$$\sqrt{13}\sin\theta - 6\cos\theta = -4.$$ [6]

(Summer 15)

a) $\tan(x+45) = 8\tan x$

use $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$

equation becomes:-

$$\frac{\tan x + \tan 45}{1 - \tan x \tan 45} = 8\tan x \qquad (\tan 45 = 1)$$

$$1 + \tan x = 8\tan x(1 - \tan x)$$

$$1 + \tan x = 8\tan x - 8\tan^2 x$$

$$8\tan^2 x - 7\tan x + 1 = 0$$

quadratic will not factorise, let $y = \tan x$

$$8y^2 - 7y + 1 = 0$$

$$y = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$y = \frac{7 \pm \sqrt{49 - 32}}{16} = \frac{7 \pm \sqrt{17}}{16}$$

$$\tan x = \frac{7 - \sqrt{17}}{16} \qquad \tan x = \frac{7 + \sqrt{17}}{16}$$

$x = 10.2°$ $x = 34.8°$

MATHSDIY

b i) $\sqrt{13}\sin\theta - 6\cos\theta = R\sin(\theta-\alpha)$

$\sqrt{13}\sin\theta - 6\cos\theta = R\sin\theta\cos\alpha - R\cos\theta\sin\alpha$

$\sqrt{13} = R\cos\alpha$ ① $\quad 6 = R\sin\alpha$ ②

① ^{2} + ② ^{2} $\Rightarrow$ $R^2(\sin^2\alpha + \cos^2\alpha) = 6^2 + (\sqrt{13})^2$

$R^2 = 49$

$R = 7$

②/① $\dfrac{R\sin\alpha}{R\cos\alpha} = \dfrac{6}{\sqrt{13}}$

$\tan\alpha = \dfrac{6}{\sqrt{13}}$

$\alpha = 59.0°$

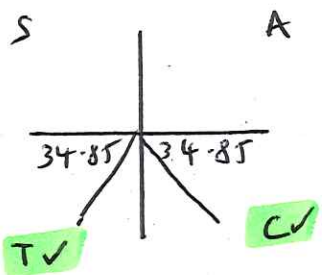
$7\sin(\theta - 59.0°)$

ii) $\sqrt{13}\sin\theta - 6\cos\theta = -4$

Equation becomes

$7\sin(\theta - 59.0°) = -4$

$\sin(\theta - 59.0°) = \dfrac{-4}{7}$



$\theta - 59.0° = -34.85°$,

$\theta - 59.0° = -34.85°, 214.85°, 325.15°$ (325.15° → too big)

$\theta = 24.15°, 273.85°$

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7.

(a) The angle x is such that $0° \leqslant x \leqslant 180°$, $x \neq 90°$.

Given that x satisfies the equation $3\tan 2x + 16\cot^2 x = 0$,

(i) show that $3\tan^3 x - 8\tan^2 x + 8 = 0$,

(ii) find all possible values of x, giving each answer in degrees, correct to one decimal place. [8]

(b) Express $24\cos\theta - 7\sin\theta$ in the form $R\cos(\theta + \alpha)$, where R and α are constants with $R > 0$ and $0° < \alpha < 90°$.

Hence, find the range of values of k for which the equation

$$24\cos\theta - 7\sin\theta = k$$

has no solutions. [5]

(Summer 16)

a)i) $3\tan 2x + 16\cot^2 x = 0$ $\left(\cot x \equiv \frac{1}{\tan x}\right)$

$3\tan 2x + \frac{16}{\tan^2 x} = 0$

Use $\tan 2x \equiv \frac{2\tan x}{1-\tan^2 x}$

$3\left(\frac{2\tan x}{1-\tan^2 x}\right) + \frac{16}{\tan^2 x} = 0$ (expand & $\times \tan^2 x$)

$\frac{6\tan^3 x}{1-\tan^2 x} + 16 = 0$

$6\tan^3 x = -16(1-\tan^2 x)$

$6\tan^3 x - 16\tan^2 x + 16 = 0$ $(\div 2)$

$3\tan^3 x - 8\tan^2 x + 8 = 0$ as required

ii) Let $y = \tan x$

equation becomes

$3y^3 - 8y^2 + 8 = 0$

Now use the factor theorem to find a factor.

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$f(y) = 3y^3 - 8y^2 + 8$

$f(1) = 3(1) - 8(1) + 8 \neq 0$

$f(-1) = 3(-1)^3 - 8(-1)^2 + 8 \neq 0$

$f(2) = 3(2)^3 - 8(2)^2 + 8 = 0$

$(y-2)$ is a factor of $3y^3 - 8y^2 + 8$

$3y^3 - 8y^2 + 8 = (y-2)(ay^2 + by + c)$

Comparing coefficients y^3:- $a = 3$

Const:- $c = -4$

y:- $-2b + c = 0$

$c = 2b$, $b = -2$

$(y-2)(3y^2 - 2y - 4)$ quadratic doesn't factorise

$y = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, $y = \frac{2 \pm \sqrt{4 + 48}}{6}$, $y = 1.5352$

$y = -0.8685$

$y = \tan x$

$\tan x = 2$ $\tan x = 1.5352$ $\tan x = -0.8685$

$x = 63.4°$ $x = 56.9°$ $x = 139.0°$

b) $24\cos\theta - 7\sin\theta = R\cos(\theta + \alpha)$

$24\cos\theta - 7\sin\theta = R\cos\theta\cos\alpha - R\sin\theta\sin\alpha$

① $24 = R\cos\alpha$ ② $7 = R\sin\alpha$

①^2 + ②^2 $24^2 + 7^2 = R^2(\sin^2\alpha + \cos^2\alpha)$

$625 = R^2$, $R = 25$

②/① $\frac{R\sin\alpha}{R\cos\alpha} = \frac{7}{24}$, $\tan\alpha = \frac{7}{24}$, $\alpha = 16.26°$

$25\cos(\theta + 16.26)$

Equation becomes $25\cos(\theta + 16.26) = k$

Now, $-1 \leq \cos(\theta + 16.26) \leq 1$

So, $-25 \leq 25\cos(\theta + 16.26) \leq 25$

$-25 \leq k \leq 25$

So No solutions if $k < -25$, $k > 25$

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8.

(a) Show that the equation

$$5\cos^2\theta + 7\sin 2\theta = 3\sin^2\theta$$

may be rewritten in the form

$$a\tan^2\theta + b\tan\theta + c = 0,$$

where a, b, c are non-zero constants whose values are to be found.
Hence, find all values of θ in the range $0° \leqslant \theta \leqslant 180°$ satisfying the equation

$$5\cos^2\theta + 7\sin 2\theta = 3\sin^2\theta.$$ [6]

(b) (i) Express $\sqrt{5}\cos\phi + \sqrt{11}\sin\phi$ in the form $R\cos(\phi - \alpha)$, where R and α are constants with $R > 0$ and $0° < \alpha < 90°$.

(ii) Use your result to part (i) to find the least value of

$$\frac{1}{\sqrt{5}\cos\phi + \sqrt{11}\sin\phi + 6}.$$

Write down a value for ϕ for which this least value occurs. [6]

(Summer 17)

a) $5\cos^2\theta + 7\sin 2\theta = 3\sin^2\theta$

use $\sin 2\theta \equiv 2\sin\theta\cos\theta$

$5\cos^2\theta + 7(2\sin\theta\cos\theta) = 3\sin^2\theta$

$5\cos^2\theta + 14\sin\theta\cos\theta - 3\sin^2\theta = 0$

$(\div \cos^2\theta)$

$5 + 14\frac{\sin\theta}{\cos\theta} - 3\frac{\sin^2\theta}{\cos^2\theta} = 0$

$\left(\tan\theta = \frac{\sin\theta}{\cos\theta}\right)$

$5 + 14\tan\theta - 3\tan^2\theta = 0$

$3\tan^2\theta - 14\tan\theta - 5 = 0$

xac	+b
-15	-14

-15, 1

$3\tan^2\theta - 15\tan\theta + \tan\theta - 5 = 0$

$3\tan\theta(\tan\theta - 5) + 1(\tan\theta - 5) = 0$

$(\tan\theta - 5)(3\tan\theta + 1) = 0$

$\tan\theta = 5$ $\quad$ $\tan\theta = -\frac{1}{3}$

$\theta = 78.7°$ $\quad$ $\theta = 161.6°$

bi) $\sqrt{5}\cos\phi + \sqrt{11}\sin\phi = R\cos(\phi - \alpha)$

$\sqrt{5}\cos\phi + \sqrt{11}\sin\phi = R\cos\phi\cos\alpha + R\sin\phi\sin\alpha$

$\sqrt{5} = R\cos\alpha$ (1) $\quad$ $\sqrt{11} = R\sin\alpha$ (2)

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$①^2 + ②^2$ $\quad (\sqrt{5})^2 + (\sqrt{11})^2 = R^2(\sin^2\alpha + \cos^2\alpha)$

$16 = R^2$ $\quad \underline{R = 4}$

$②/①$ $\quad \dfrac{R\sin\alpha}{R\cos\alpha} = \dfrac{\sqrt{11}}{\sqrt{5}}$

$\tan\alpha = \dfrac{\sqrt{11}}{\sqrt{5}}$

$\underline{\alpha = 56.01^\circ}$

$\underline{4\cos(\phi - 56.01^\circ)}$

Least value occurs when the denominator is as big as possible.

$$\frac{1}{\sqrt{5}\cos\phi + \sqrt{11}\sin\phi + 6}$$

becomes, $\dfrac{1}{4\cos(\phi - 56.01) + 6}$

1 is the max value of $\cos(\phi - 56.01)$

Least value $= \dfrac{1}{4+6} = \underline{\dfrac{1}{10}}$

$\cos(\phi - 56.01) = 1$

$\phi - 56.01 = 0$

$\underline{\phi = 56.01^\circ}$

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