WJEC: A2 Unit 3 Pure Mathematics B

Sample Paper 2 hour 30 minutes

PAPER WITH TEACHER'S SOLUTIONS

Total marks available: 120

Reminder: *Sufficient working must be shown to demonstrate the **mathematical** method employed.*

1. Find a small positive value of x which is an approximate solution of the equation.

$$\cos x - 4\sin x = x^2.$$ [4]

Remember your small angle approximations:-

- $\sin\theta \approx \theta$
- $\cos\theta \approx 1 - \frac{\theta^2}{2}$
- $\tan\theta \approx \theta$

$\therefore$ $\cos x - 4\sin x = x^2$ becomes

$1 - \frac{x^2}{2} - 4x = x^2$

$0 = x^2 + \frac{x^2}{2} + 4x - 1 \quad \times 2$

$0 = 2x^2 + x^2 + 8x - 2$

$0 = 3x^2 + 8x - 2$ won't factorise, use quadratic formula

$x = \frac{-8 \pm \sqrt{64 - 4(3)(-2)}}{2(3)} = \frac{-8 \pm \sqrt{88}}{6}$

$x = 0.2301$, $x = -2.8968$

require a small, positive x

$x = 0.2301$

2. Air is pumped into a spherical balloon at the rate of 250 cm^3 per second. When the radius of the balloon is 15 cm, calculate the rate at which the radius is increasing, giving your answer to three decimal places [3]

Volume of sphere $= \frac{4}{3}\pi r^3$

Rate of increase of air in balloon $= 250\ cm^3/second$

(look at the units) $\therefore \frac{dV}{dt} = 250$ (1)

Differentiation gives $\frac{dV}{dr} = 3\left(\frac{4}{3}\right)\pi r^2 = 4\pi r^2$ (2)

And, $\frac{dV}{dt} = \frac{dV}{dr} \cdot \frac{dr}{dt}$, substitute in (1) and (2)

$$250 = 4\pi r^2 \cdot \frac{dr}{dt}$$

$$\frac{dr}{dt} = \frac{250}{4\pi r^2}$$

Require $\frac{dr}{dt}$ at $r = 15$

$$\Rightarrow \frac{250}{4\pi(15)^2}$$

$$= 0.088\ cm/second \quad (3dp)$$

3. (a) Sketch the graph of $y = x^2 + 6x + 13$, identifying the stationary point. [2]

(b) The function f is defined by $f(x) = x^2 + 6x + 13$ with domain (a, b).

(i) Explain why f^{-1} does not exist when $a = -10$ and $b = 10$. [1]

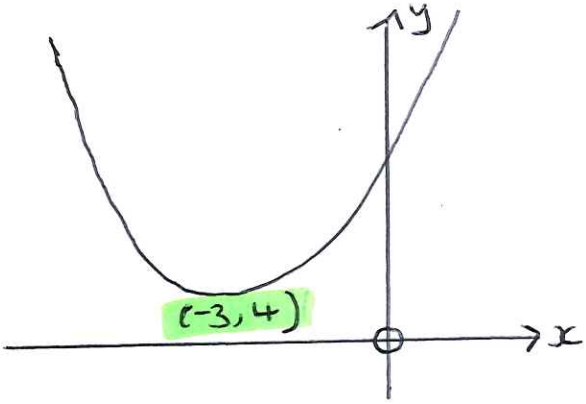
(ii) Write down a value of a and a value of b for which the inverse of f does exist and derive an expression for $f^{-1}(x)$. [5]

a) use completing the square to find the minimum

$y = (x+3)^2 - 9 + 13$

$y = (x+3)^2 + 4$

minimum = $(-3, 4)$



bi) f^{-1} would not exist in domain $(-10, 10)$ because f is not a one to one function.

ii) $(-3, \infty)$ (several answers are possible)

Let $y = x^2 + 6x + 13$

$y = (x+3)^2 + 4$

$y - 4 = (x+3)^2$

$\pm\sqrt{y-4} = x + 3$

$x = -3 \pm \sqrt{y-4}$

Now, check the domain, $-3 - \sqrt{y-4}$ wouldn't be in the domain. Therefore $x = -3 + \sqrt{y-4}$

$\therefore f^{-1}(x) = -3 + \sqrt{x-4}$

4. (a) Expand $(1-x)^{-\frac{1}{2}}$ in ascending power of x as far as the term in x^2. State the range of x for which the expansion is valid. [2]

(b) By taking $x=\frac{1}{10}$, find an approximation for $\sqrt{10}$ in the form $\frac{a}{b}$, where a and b are to be determined. [2]

a) Formula booklet page 2

$(1+x)^n = 1+nx+\frac{n(n-1)}{2}x^2+\cdots$ valid $|x|<1$, $n\in\mathbb{R}$

Substitute into this $n=-\frac{1}{2}$ and $x=(-x)$

$$(1-x)^{-1/2} = 1-\tfrac{1}{2}(-x)+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)(-x)^2}{2}+\cdots$$

$$= 1+\frac{x}{2}+\frac{3x^2}{8}+\cdots$$ valid $|x|<1$

b) Substitute into this $x=\frac{1}{10}$

$$\left(1-\tfrac{1}{10}\right)^{-1/2} = 1+\frac{\left(\frac{1}{10}\right)}{2}+\frac{3\left(\frac{1}{10}\right)^2}{8}+\cdots$$

$$\left(\frac{9}{10}\right)^{-1/2} = 1+\frac{1}{20}+\frac{3}{800}+\cdots$$

$$1\Big/\sqrt{\frac{9}{10}} \approx \frac{843}{800}$$

$$\frac{\sqrt{10}}{3} \approx \frac{843}{800}$$

$$\sqrt{10} \approx \frac{3\times 843}{800} = \frac{2529}{800}$$

5. Aled decides to invest £1000 in a savings scheme on the first day of each year. The scheme pays 8% compound interest per annum, and interest is added on the last day of each year. The amount of savings, in pounds, at the end of the third year is given by

$$1000\times1\cdot08+1000\times1\cdot08^2+1000\times1\cdot08^3$$

Calculate, to the nearest pound, the amount of savings at the end of thirty years. [5]

At the end of the thirty years, Alex's amount of savings will be given by

$$1000(1\cdot08)+1000(1\cdot08)^2+\cdots+1000(1\cdot08)^{30}$$

This is a geometric progression where:-

$a=1000(1\cdot08)$, $n=30$, $r=1\cdot08$ & we require S_{30}

$S_n=\frac{a(1-r^n)}{1-r}$ formula booklet page 2

$$S_{30}=\frac{1000(1\cdot08)(1-1\cdot08^{30})}{1-1\cdot08}$$

$$=122345.868$$

Alex's savings will be £122,346 after 30 years (nearest £)

6. The lengths of the sides of a fifteen-sided plane figure form an arithmetic sequence. The perimeter of the figure is 270 cm and the length of the largest side is eight times that of the smallest side. Find the length of the smallest side. [4]

Perimeter = 270

$S_n = \frac{1}{2} n(a+L)$ page 2 formula booklet

length of smallest side = a

length of largest side, $L = 8a$

$n = 15$

$S_{15} = 270$

$270 = \frac{1}{2} \times 15(a+8a)$

$\frac{270 \times 2}{15} = 9a$

$36 = 9a$, $a = 4cm$

Length of smallest side = 4cm

7. The curve $y = ax^4 + bx^3 + 18x^2$ has a point of inflection at (1, 11).

(a) Show that $2a + b + 6 = 0$. [2]

(b) Find the values of the constants a and b and show that the curve has another point of inflection at (3, 27). [8]

(c) Sketch the curve, identifying all the stationary points including their nature. [6]

a) Points of inflection can occur when $\frac{d^2y}{dx^2} = 0$

$y = ax^4 + bx^3 + 18x^2$

$\frac{dy}{dx} = 4ax^3 + 3bx^2 + 36x$

$\frac{d^2y}{dx^2} = 12ax^2 + 6bx + 36$

∴ For a point of inflection at (1, 11), we can substitute in $x = 1$ and $\frac{d^2y}{dx^2} = 0$

$0 = 12a(1)^2 + 6b(1) + 36$

$12a + 6b + 36 = 0 \quad \div 6$

① $2a + b + 6 = 0$ as required

b) $x = 1$, $y = 11$ into equation of curve

$a(1)^4 + b(1)^3 + 18(1)^2 = 11$

$a + b + 18 = 11$

$b = -7 - a$ ②

② in ① $2a - 7 - a + 6 = 0$

$a = 1$, in ② $b = -7 - 1$, $b = -8$

$\frac{d^2y}{dx^2} = 12(1)x^2 + 6(-8)x + 36 = 12x^2 - 48x + 36$

$= 12(x^2 - 4x + 3)$

$= 12(x-1)(x-3) = 0$ ← point of inflection

$\therefore x = 3$, is a potential further point of inflection. Check this by looking for a change in sign of $\frac{d^2y}{dx^2}$ as x passes through 3.

at $x = 2.9$ $\frac{d^2y}{dx^2} = 12(2.9)^2 - 48(2.9) + 36 < 0$

at $x = 3.1$ $\frac{d^2y}{dx^2} = 12(3.1)^2 - 48(3.1) + 36 > 0$ ∴ change of sign

$y = (1)(3^4) - 8(3^3) + 18(3)^2 = 27$ Point of inflection at (3, 27)

c) Check for further stationary values

$\frac{dy}{dx} = 4x^3 - 24x^2 + 36x$

Stationary values occur at $\frac{dy}{dx} = 0$

$4x^3 - 24x^2 + 36x = 0$

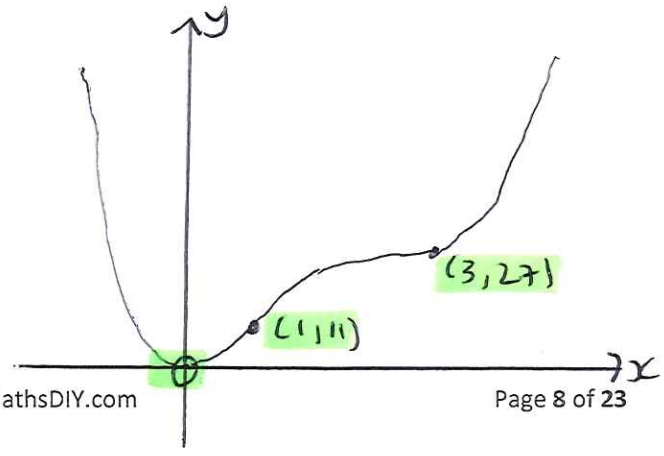
$x^3 - 6x^2 + 9x = 0$

$x(x^2 - 6x + 9) = 0$

$x(x-3)(x-3) = 0$

$x = 0$, P.O.I at $x = 3$

at $x = 0$, $\frac{d^2y}{dx^2} = 36 > 0$, MINIMUM



8. (a) Integrate

(i) e^{-3x+5} [2]

(ii) $x^2 \ln x$ [4]

(b) Use an appropriate substitution to show that

$$\int_0^{\frac{1}{2}} \frac{x^2}{\sqrt{1-x^2}}\,dx = \frac{\pi}{12} - \frac{\sqrt{3}}{8}.$$ [8]

a) i) $\int e^{-3x+5}\,dx = \frac{e^{-3x+5}}{-3} + c = -\frac{1}{3}e^{-3x+5} + c$

ii) $\int x^2 \ln x\,dx$ Using integration by parts

$\frac{dv}{dx}$ u

$u = \ln x$ $\frac{du}{dx} = \frac{1}{x}$ $\frac{dv}{dx} = x^2$ $v = \frac{x^3}{3}$

$$\int u\frac{dv}{dx} = uv - \int v\frac{du}{dx}\,dx$$

$$= \ln x\left(\frac{x^3}{3}\right) - \int \frac{x^3}{3}\cdot\frac{1}{x}\,dx$$

$$= \frac{x^3}{3}\ln x - \frac{1}{3}\int x^2\,dx$$

$$= \frac{x^3}{3}\ln x - \frac{1}{3}\left(\frac{x^3}{3}\right) + c$$

$$= \frac{x^3}{3}\ln x - \frac{x^3}{9} + c$$

b) $\int_0^{1/2} \frac{x^2}{\sqrt{1-x^2}}\,dx$

Let $x = \sin\theta$

$\frac{dx}{d\theta} = \cos\theta$

$dx = \cos\theta\, d\theta$

$x^2 = \sin^2\theta$

$\sqrt{1-x^2} = \sqrt{1-\sin^2\theta}$

but $\cos^2\theta \equiv 1-\sin^2\theta$

$\therefore \sqrt{1-x^2} = \sqrt{\cos^2\theta} = \cos\theta$

when $x=0$, $\sin\theta = 0$

$\theta = 0$

when $x = \frac{1}{2}$, $\sin\theta = \frac{1}{2}$

$\theta = \pi/6$

Integral becomes :-

$\int_0^{\pi/6} \frac{\sin^2\theta}{\cos\theta} \cdot \cos\theta\, d\theta$

$= \int_0^{\pi/6} \sin^2\theta\, d\theta$

Now, $\cos 2\theta = \cos^2\theta - \sin^2\theta$

$\cos 2\theta = (1-\sin^2\theta) - \sin^2\theta$

$\cos 2\theta - 1 = -2\sin^2\theta$

$2\sin^2\theta = 1 - \cos 2\theta$

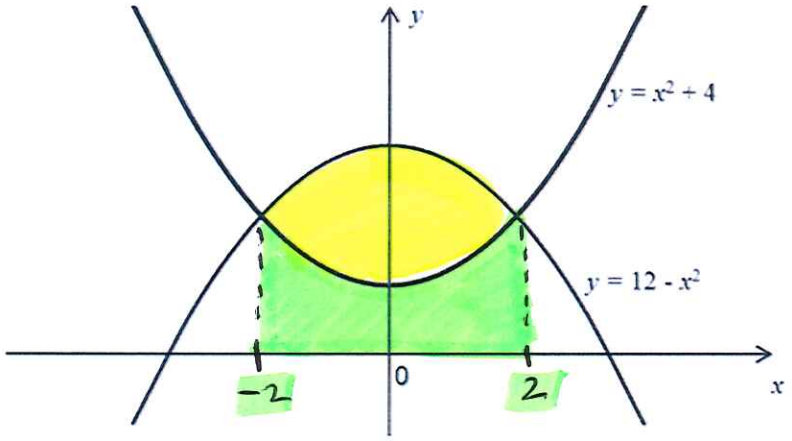
$\sin^2\theta = \frac{1-\cos 2\theta}{2}$

$= \int_0^{\pi/6} \frac{1-\cos 2\theta}{2}\, d\theta$

$= \int_0^{\pi/6} \frac{1}{2}\, d\theta - \frac{1}{2}\int_0^{\pi/6} \cos 2\theta\, d\theta = \left[\frac{\theta}{2} - \frac{\sin 2\theta}{4}\right]_0^{\pi/6}$

$= \left(\frac{\pi}{12} - \frac{\sin \pi/3}{4}\right) - (0-0) = \frac{\pi}{12} - \frac{\sqrt{3}}{8}$

9.



The diagram above shows a sketch of the curves $y = x^2 + 4$ and $y = 12 - x^2$.

Find the area of the region bounded by the two curves. [6]

Need to find coordinates of points of intersection first:-

$x^2 + 4 = 12 - x^2$

$2x^2 = 8$

$x^2 = 4$

$x = 2$, $x = -2$

Required Area = $\int_{-2}^{2} (12 - x^2)\,dx - \int_{-2}^{2} (x^2 + 4)\,dx$

This area is shaded green above

Yellow shading

$= \int_{-2}^{2} (12 - x^2 - x^2 - 4)\,dx$

$= \int_{-2}^{2} (8 - 2x^2)\,dx$

$= \left[8x - \frac{2x^3}{3} \right]_{-2}^{2}$

$$= \left(8(2) - \frac{2(2)^3}{3}\right) - \left(8(-2) - \frac{2(-2)^3}{3}\right)$$

$$= 16 - \frac{16}{3} + 16 - \frac{16}{3}$$

$$= \frac{64}{3}$$

10. The equation

$$1+5x-x^4=0$$

has a positive root α.

(a) Show that α lies between 1 and 2. [2]

(b) Use the iterative sequence based on the arrangement

$$x=\sqrt[4]{1+5x}$$

with starting value 1.5 to find α correct to two decimal places. [3]

(c) Use the Newton-Raphson method to find α correct to six decimal places. [6]

a) Let $f(x)=1+5x-x^4$

$f(1)=1+5-1=5$

$f(2)=1+10-16=-5$

Change of sign, $\therefore 1<\alpha<2$

b) $x_{n+1}=\sqrt[4]{1+5x_n}$

$x_0=1.5$

$x_1=1.707476485$

$x_2=1.757346089$

$x_3=1.768721303$

$x_4=1.771285475$

$\alpha=1.77$ (2dp)

c) $f(x)=1+5x-x^4$

$f'(x)=5-4x^3$

and, from formula booklet $x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}$

This gives, $x_{n+1}=x_n-\frac{1+5x_n-x_n^4}{5-4x_n^3}$

$x_0 = 1.5$

$x_1 = 1.904411765$

$x_2 = 1.788115338$

$x_3 = 1.772305155$

$x_4 = 1.772029085$

$x_5 = 1.772029002$

$x_6 = 1.772029002$

$\alpha = 1.772029$ (6dp)

11. (a) The curve C is given by the equation

$$x^4 + x^2y + y^2 = 13.$$

Find the value of $\frac{dy}{dx}$ at the point (-1, 3). [4]

(b) Show that the equation of the normal to the curve $y^2 = 4x$ at the point $P(p^2, 2p)$ is

$$y + px = 2p + p^3.$$

Given that $p \neq 0$ and that the normal at P cuts the x-axis at $B(b, 0)$, show that $b > 2$. [7]

a) Implicit differentiation gives

$$4x^3 + x^2\frac{dy}{dx} + 2xy + 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx}(x^2 + 2y) = -4x^3 - 2xy$$

$$\frac{dy}{dx} = \frac{-4x^3 - 2xy}{x^2 + 2y}$$

at $(-1, 3)$ $\frac{dy}{dx} = \frac{-4(-1)^3 - 2(-1)(3)}{(-1)^2 + 2(3)} = \frac{10}{7}$

b) $x = p^2 \quad y = 2p$

$\frac{dx}{dp} = 2p \quad \frac{dy}{dp} = 2$

$$\frac{dy}{dx} = \frac{dy/dp}{dx/dp} = \frac{2}{2p} = \frac{1}{p}$$

$\therefore$ Gradient of tangent to curve at P is $\frac{1}{p}$

$\therefore$ Gradient of normal to curve at P is $-p$

Now use $(y-y_1)=m(x-x_1)$

$y-2p=-p(x-p^2)$

$y+px=2p+p^3$ as required

Substitute $(b,0)$ into above

$pb=2p+p^3$ given $p\neq 0$

$(\div p)$ $b=2+p^2$

because $p\neq 0$, $p^2>0$ $\therefore$ $b>2$

12. (a) Differentiate $\cos x$ from first principles. [5]

(b) Differentiate the following with respect to x, simplifying your answer as far as possible.

(i) $\dfrac{3x^2}{x^3+1}$ [2]

(ii) $x^3 \tan 3x$ [2]

$f(x) = \cos x$

$f'(x) = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

$f'(x) = \lim_{h \to 0} \left(\dfrac{\cos(x+h) - \cos x}{h} \right)$

now, use $\cos(x+h) = \cos x \cos h - \sin x \sin h$

$f'(x) = \lim_{h \to 0} \left(\dfrac{\cos x \cos h - \sin x \sin h - \cos x}{h} \right)$

now, using our small angle approximation knowledge,

as $h \to 0$, $\cos h \approx 1 - \dfrac{h^2}{2}$ and $\sin h \approx h$

$\Rightarrow \quad f'(x) = \lim_{h \to 0} \left(\dfrac{\cos x \left(1 - \frac{h^2}{2}\right) - \sin x (h) - \cos x}{h} \right)$

$= \lim_{h \to 0} \left(\dfrac{\cos x - \frac{h^2}{2} \cos x - h \sin x - \cos x}{h} \right)$

$= \lim_{h \to 0} \dfrac{h\left(-\frac{h}{2} \cos x - \sin x\right)}{h}$

as $h \to 0 \quad \frac{h}{2} \cos x \to 0$

$f'(x) = -\sin x$

b i) $y = \frac{3x^2}{x^3+1}$ → u → v Quotient Rule

$$y = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$u = 3x^2$ $\quad v = x^3+1$

$\frac{du}{dx} = 6x$ $\quad \frac{dv}{dx} = 3x^2$

$$\frac{dy}{dx} = \frac{(x^3+1)(6x) - (3x^2)(3x^2)}{(x^3+1)^2}$$

$$= \frac{6x^4 + 6x - 9x^4}{(x^3+1)^2}$$

$$= \frac{-3x^4 + 6x}{(x^3+1)^2}$$

ii) $y = x^3 \tan 3x$ → u → v Product Rule

$$\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$$

$u = x^3$ $\quad v = \tan 3x$

$\frac{du}{dx} = 3x^2$ $\quad \frac{dv}{dx} = 3\sec^2 3x$

$$\frac{dy}{dx} = (\tan 3x)(3x^2) + x^3(3\sec^2 3x)$$

$$= 3x^2(\tan 3x + x\sec^2 3x)$$

13. (a) Solve the equation

$$\operatorname{cosec}^2 x + \cot^2 x = 5$$

for $0^\circ \le x \le 360^\circ$. [5]

(b) (i) Express $4\sin\theta + 3\cos\theta$ in the form $R\sin(\theta + \alpha)$, where $R > 0$ and $0^\circ \le \alpha \le 90^\circ$. [4]

(ii) Solve the equation

$$4\sin\theta + 3\cos\theta = 2$$

for $0^\circ \le \theta \le 360^\circ$, giving your answer correct to the nearest degree. [3]

a) Rewrite equation using $\operatorname{cosec}^2 x \equiv 1 + \cot^2 x$

Equation becomes $(1 + \cot^2 x) + \cot^2 x = 5$

$2\cot^2 x = 4$

$\cot^2 x = 2$

$\frac{1}{\tan^2 x} = 2$

$\frac{1}{2} = \tan^2 x$

$\tan x = \pm \frac{1}{\sqrt{2}}$

$x = 35.3^\circ, 215.3^\circ, 144.7^\circ, 324.7^\circ$

(remember, no need for CAST diagram as tan repeats every 180°)

b) i) $R\sin(\theta + \alpha) = R\sin\theta\cos\alpha + R\cos\theta\sin\alpha$

$4\sin\theta + 3\cos\theta = R\sin\theta\cos\alpha + R\cos\theta\sin\alpha$

Equating $\sin\theta$ terms $4 = R\cos\alpha$ (1)

Equating $\cos\theta$ terms $3 = R\sin\alpha$ (2)

$(1)^2 + (2)^2$ gives $R^2\cos^2\alpha + R^2\sin^2\alpha = 4^2 + 3^2$

$R^2(\sin^2\alpha + \cos^2\alpha) = 25$

and $\sin^2\alpha + \cos^2\alpha \equiv 1$

$R^2 = 25$, $R = 5$

(2)/(1) $\frac{R\sin\alpha}{R\cos\alpha} = \frac{3}{4}$, $\tan\alpha = \frac{3}{4}$

$\alpha = 36.87^\circ$ $= 5\sin(\theta + 36.87^\circ)$

ii) $4\sin\theta + 3\cos\theta = 2$

$5\sin(\theta + 36.87) = 2$

$\sin(\theta + 36.87) = \frac{2}{5}$

$\theta + 36.87 = 23.58°, 156.42, 383.58$

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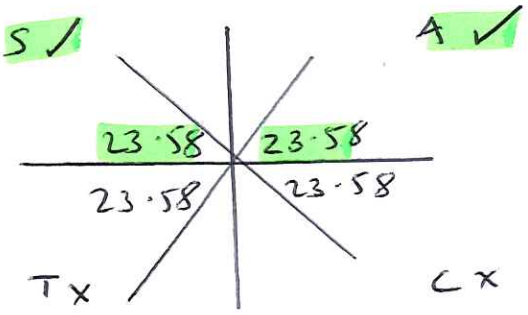

$\theta = 156.42 - 36.87 = 119.55°$

$\theta = 383.58 - 36.87 = 346.71°$

$\theta = 120°, 347°$ (to nearest degree)

14. (a) A cylindrical water tank has base area 4 m^2. The depth of the water at time t seconds is h metres. Water is poured in at the rate 0.004 m^3 per second. Water leaks from a hole in the bottom at a rate of $0.0008h$ m^3 per second. Show that

$$5000\frac{dh}{dt} \equiv 5 - h. \quad [2]$$

[Hint: the volume, V, of the cylindrical water tank is given by $V = 4h$.]

(b) Given that the tank is empty initially, find h in terms of t. [7]

(c) Find the depth of the water in the tank when $t = 3600$ s, giving your answer correct to 2 decimal places. [1]

a) Volume of cylinder $= \pi r^2 h$, Area of base $= 4m^2$

$V = \pi r^2 h = 4h$

$$\frac{dV}{dh} = 4$$

$$\frac{dV}{dt} = 0.004 - 0.0008h$$

$$\frac{dV}{dt} = \frac{dV}{dh} \cdot \frac{dh}{dt}$$

$$0.004 - 0.0008h = 4\frac{dh}{dt}$$

$$\frac{0.004 - 0.0008h}{4} = \frac{dh}{dt}$$

$$\frac{dh}{dt} = 0.001 - 0.0002h \quad (\times 5000)$$

$$5000\frac{dh}{dt} = 5 - h$$ as required

b) $$\frac{5000}{5-h} \cdot \frac{dh}{dt} = 1$$

Seperating the variables and integrating gives

$$\int \frac{5000}{5-h}\,dh = \int dt$$

$$5000\,\frac{\ln|5-h|}{-1} = t + c$$

$$-5000\ln|5-h| = t + c \quad (1)$$

given $h=0$ when $t=0$

$$-5000\ln 5 = c$$

in (1), gives $-5000\ln|5-h| = t - 5000\ln 5$

$$5000(\ln 5 - \ln|5-h|) = t$$

$$5000\ln\left(\frac{5}{5-h}\right) = t$$

Now rewrite as a power

$$\left(\frac{5}{5-h}\right) = e^{\frac{t}{5000}}$$

$$\frac{5}{e^{t/5000}} = 5-h$$

$$h = 5 - 5e^{\frac{-t}{5000}}$$

c) at $t = 3600\,s$

$$h = 5 - 5e^{-\frac{3600}{5000}}$$

$$h = 2.57\,m \quad (2dp)$$

15. Prove by contradiction the following proposition.

When x is real and positive,

$$4x + \frac{9}{x} \geq 12.$$

The first line of the proof is given below.

Assume that there is a positive and a real value of x such that

$$4x + \frac{9}{x} < 12. \qquad [3]$$

Assume that there is a real, positive x such that

$$4x + \frac{9}{x} < 12 \qquad \times x \ (x)$$

$$4x^2 + 9 < 12x$$

$$4x^2 - 12x + 9 < 0$$

$$4x^2 - 6x - 6x + 9 < 0$$

$$2x(2x-3) - 3(2x-3) < 0$$

$$(2x-3)^2 < 0$$

Impossible when x is real

Contradiction, assumption is false

$$\therefore \quad 4x + \frac{9}{x} \geq 12$$

x	$+6$
ac	
36	-12

$-6, -6$