# WJEC – AS unit 1 Pure Mathematics A

## Sample Assessment Materials

## PAPER WITH TEACHER'S SOLUTIONS

**Total marks available: 120**

**Reminder**: *Sufficient working must be shown to demonstrate the **mathematical** method employed.*

1. The circle $C$ has centre $A$ and equation

$$x^2 + y^2 - 2x + 6y - 15 = 0.$$

(a) Find the coordinates of $A$ and the radius of $C$. [3]

(b) The point $P$ has coordinates $(4, -7)$ and lies on $C$. Find the equation of the tangent to $C$ at $P$. [4]

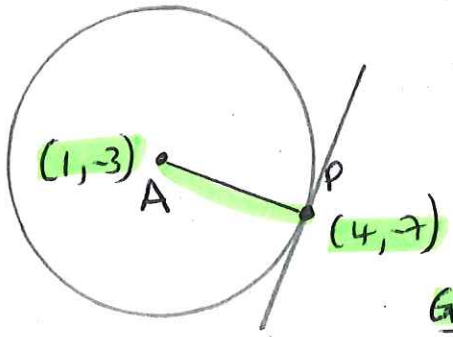
a) Rewrite as $x^2 - 2x + y^2 + 6y = 15$

$(x-1)^2 - 1 + (y+3)^2 - 9 = 15$

$(x-1)^2 + (y+3)^2 = 5^2$

$A = (1, -3)$ radius $= 5$

b)




Note: gradient of tangent is perpendicular to the gradient of the radius.

Gradient of radius $AP = \frac{-7 - -3}{4 - 1} = \frac{-4}{3}$

$\therefore$ Gradient of tangent $= \frac{3}{4}$

and $(4, -7)$ lies on tangent $\therefore (y - y_1) = m(x - x_1)$

$y - -7 = \frac{3}{4}x - \frac{3}{4}(4)$

$y + 7 = \frac{3}{4}x - 3$

$4y - 3x + 40 = 0$

2. Find all values of $\theta$ between 0° and 360° satisfying

$$7\sin^2\theta + 1 = 3\cos^2\theta - \sin\theta.$$ [6]

$7\sin^2\theta + \cos^2\theta \equiv 1$

$\therefore \cos^2\theta \equiv 1 - \sin^2\theta$

substitute this into equation

$7\sin^2\theta + 1 = 3(1 - \sin^2\theta) - \sin\theta$

$10\sin^2\theta + \sin\theta - 2 = 0$

| x ac | +b |
|---|---|
| -20 | 1 |

-4, 5

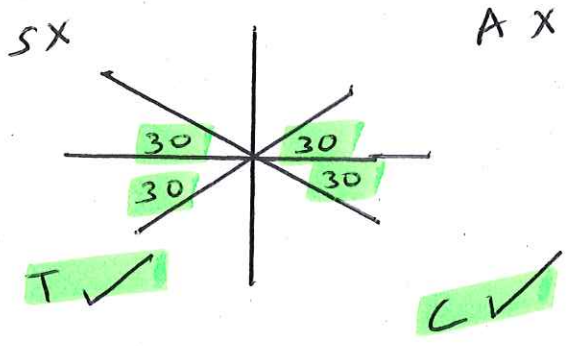
$10\sin^2\theta + 5\sin\theta - 4\sin\theta - 2 = 0$

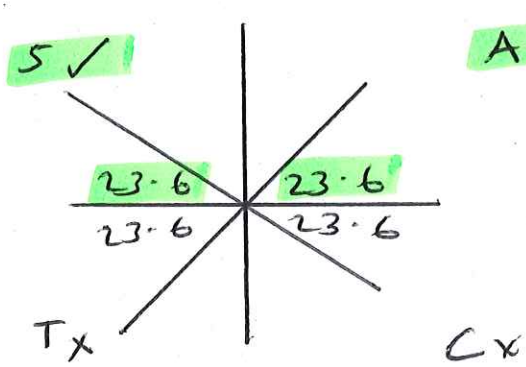
$5\sin\theta(2\sin\theta + 1) - 2(2\sin\theta + 1) = 0$

$(2\sin\theta + 1)(5\sin\theta - 2) = 0$

either $\sin\theta = -\frac{1}{2}$ or $\sin\theta = \frac{2}{5}$

$\theta = -30°$ $\theta = 23.6°$




$\theta = 23.6°, 156.4°, 210°, 330°$

3. Given that $y = x^3$, find $\frac{dy}{dx}$ from first principles. [6]

$y = x^3$

$(y+\delta y) = (x+\delta x)^3$

$y+\delta y = (x+\delta x)(x+\delta x)(x+\delta x)$

$y+\delta y = (x+\delta x)[x^2+2x\delta x+(\delta x)^2]$

$y+\delta y = x^3+2x^2\delta x+x(\delta x)^2+x^2\delta x+2x(\delta x)^2+(\delta x)^3$

$y+\delta y = x^3+3x^2\delta x+3x(\delta x)^2+(\delta x)^3$

subtracting y from both sides gives

$\delta y = 3x^2\delta x+3x(\delta x)^2+(\delta x)^3$

dividing by $\delta x$

$\frac{\delta y}{\delta x} = 3x^2+3x\delta x+(\delta x)^2$

now, let $\delta x \to 0$

$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = 3x^2$

Note:- you must state every single step to get all of the marks.

4. The cubic polynomial $f(x)$ is given by $f(x) = 2x^3 + ax^2 + bx + c$, where $a$, $b$, $c$ are constants. The graph of $f(x)$ intersects the $x$-axis at the points with coordinates $(-3, 0)$, $(2{\cdot}5, 0)$ and $(4, 0)$. Find the coordinates of the point where the graph of $f(x)$ intersects the $y$-axis. [5]

The graph intersects the x axis at
$(-3, 0)$
$(2.5, 0)$
$(4, 0)$
This would correspond to the following factors
$(x+3)$
$(x-2.5)$
$(x-4)$
However, if we multiplied these together
$(x+3)(x-2.5)(x+4)$
We wouldn't get the '$2x^3$' term we need
Therefore, rewrite $(x-2.5)$ as $(2x-5)$

$$f(x) = (2x-5)(x+3)(x-4)$$

Require co-ordinates where $f(x)$ intersects with y axis. At this point $x=0$

$\therefore \quad f(x) = (0-5)(0+3)(0-4)$

$$f(x) = 60$$

$f(x)$ intersects the y-axis at $(0, 60)$

5. The points $A(0, 2)$, $B(-2, 8)$, $C(20, 12)$ are the vertices of the triangle $ABC$. The point $D$ is the mid-point of $AB$.

(a) Show that $CD$ is perpendicular to $AB$. [6]

(b) Find the exact value of $\tan C\hat{A}B$. [5]

(c) Write down the geometrical name for the triangle $ABC$. [1]

a) If D is the midpoint of AB, use $D = \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$

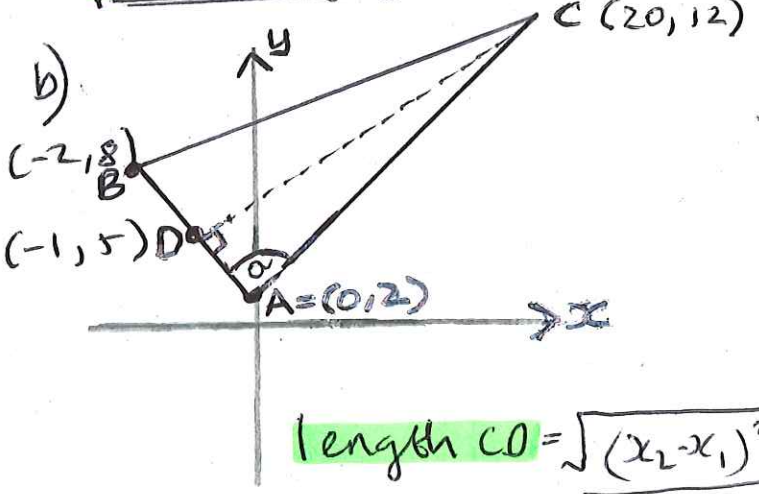
$D = (-1, 5)$

gradient of AB $= \frac{y_2-y_1}{x_2-x_1} = \frac{2-8}{0--2} = \frac{-6}{2} = -3$

gradient of CD $= \frac{12-5}{20--1} = \frac{7}{21} = \frac{1}{3}$

product of gradients $= -3 \times \frac{1}{3} = -1$ ∴ perpendicular

b)




$\tan C\hat{A}B = \tan C\hat{A}D$

$= \frac{\text{opp}}{\text{adj}}$

$= \frac{CD}{AD}$

length $CD = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}$

$= \sqrt{21^2 + 7^2} = 7\sqrt{10}$

length $AD = \sqrt{1^2+3^2} = \sqrt{10}$

$\tan C\hat{A}B = \frac{7\sqrt{10}}{\sqrt{10}} = 7$

c) isosceles

6. In each of the two statements below, $c$ and $d$ are real numbers. One of the statements is true while the other is false.

A Given that $(2c+1)^2 = (2d+1)^2$, then $c = d$.
B Given that $(2c+1)^3 = (2d+1)^3$, then $c = d$.

(*a*) Identify the statement which is false. Find a counter example to show that this statement is in fact false.

(*b*) Identify the statement which is true. Give a proof to show that this statement is in fact true. [5]

a) A is false

Conside 81, $81 = 9^2$ and $81 = (-9)^2$

So, if $(2c+1)^2 = (2d+1)^2 = 81$

then $(-9)^2 = (9)^2 = 81$

and $c = -5$ and $d = 4$ $c \neq d$

b) B is true

Any real number has a unique cube root

$(2c+1)^3 = (2d+1)^3$

$\Rightarrow$

$2c+1 = 2d+1$

$\Rightarrow$ $c = d$

7. Figure 1 shows a sketch of the graph of $y = f(x)$. The graph has a minimum point at $(-3, -4)$ and intersects the $x$-axis at the points $(-8, 0)$ and $(2, 0)$.

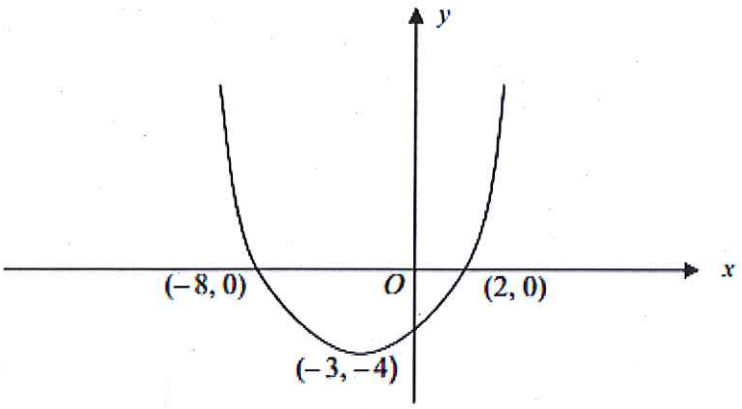



**Figure 1**

(*a*) Sketch the graph of $y = f(x + 3)$, indicating the coordinates of the stationary point and the coordinates of the points of intersection of the graph with the $x$-axis. [3]

(*b*) Figure 2 shows a sketch of the graph having **one** of the following equations with an appropriate value of either $p$, $q$ or $r$.

$y = f(px)$, where $p$ is a constant
$y = f(x) + q$, where $q$ is a constant
$y = rf(x)$, where $r$ is a constant

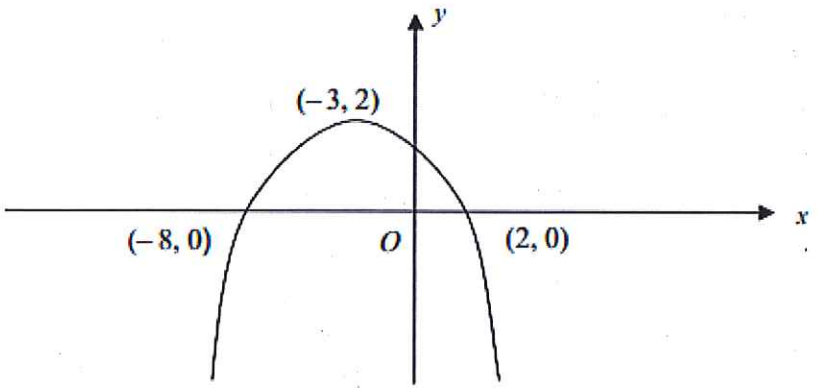
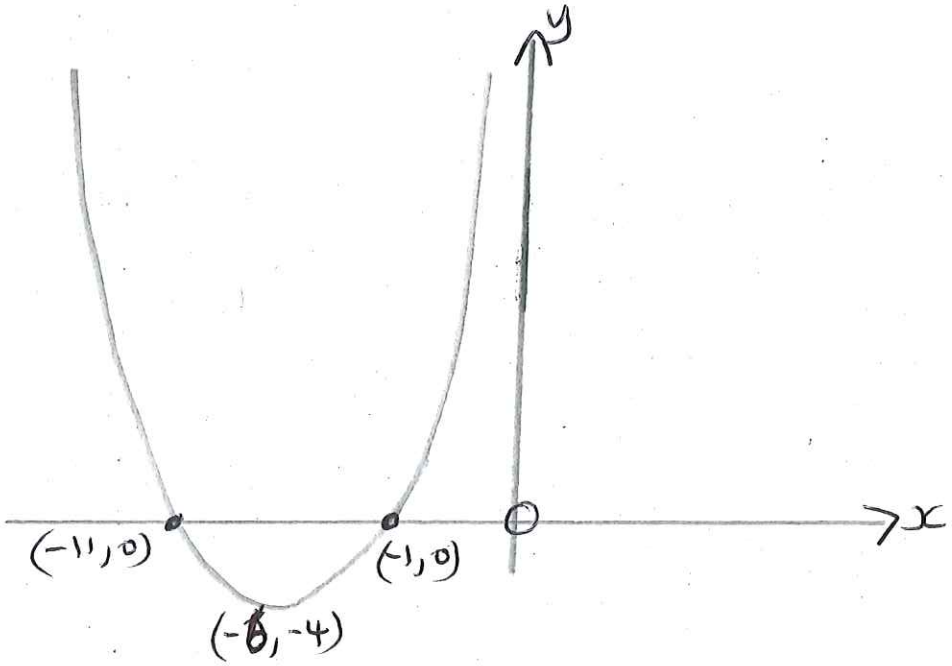



**Figure 2**

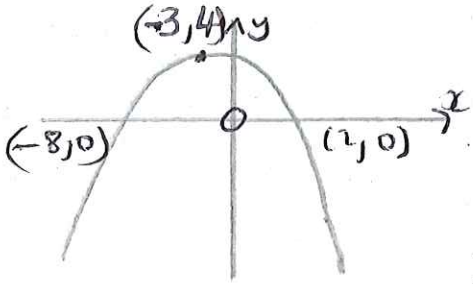
Write down the equation of the graph sketched in Figure 2, together with the value of the corresponding constant. [2]

a) $y = f(x+3)$

= translation, across x axis, 3 to the left




b) Consider $-f(x)$ ie reflected across x axis




The intercepts are ok, the shape is ok, but the maximum is wrong.

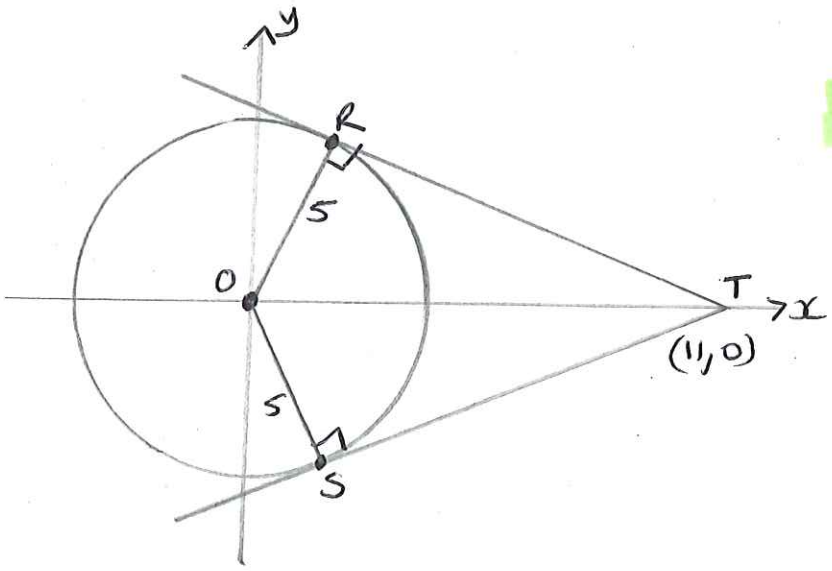
Instead of $-f(x)$ we need $-\frac{1}{2} f(x)$. But by thinking of $-f(x)$ first we can see the extra step needed (as we need stretch $\times \frac{1}{2}$ along y axis)

$y = -\frac{1}{2} f(x)$ , $r = -\frac{1}{2}$

8. The circle $C$ has radius 5 and its centre is the origin.
The point $T$ has coordinates (11, 0).
The tangents from $T$ to the circle $C$ touch $C$ at the points $R$ and $S$.

(a) Write down the geometrical name for the quadrilateral $ORTS$. [1]

(b) Find the exact value of the area of the quadrilateral $ORTS$. Give your answer in its simplest form. [5]




a) Kite

b) Recall angle between tangent and radius = 90°

Use pythagoras for length RT :- $OT^2 = OR^2 + RT^2$

$11^2 = 5^2 + RT^2$, $RT = 4\sqrt{6}$

Area of quadrilateral

= 2 × area of triangle

$= 2 \times \frac{b \times h}{2} = 5 \times 4\sqrt{6} = 20\sqrt{6}$

9. The quadratic equation $4x^2 - 12x + m = 0$, where $m$ is a positive constant, has **two distinct** real roots.
Show that the quadratic equation $3x^2 + mx + 7 = 0$ has **no** real roots. [7]

We are told that $4x^2 - 12x + m = 0$
has two distinct roots
$\therefore b^2 - 4ac > 0$
$(-12)^2 - 4(4)m > 0$
$144 > 16m$
$9 > m$ $\quad m < 9$

Now consider $3x^2 + mx + 7 = 0$
$b^2 - 4ac$
$= m^2 - 4(3)(7)$
$= m^2 - 84$

Now, $m < 9$ therefore $m^2 < 81$

so $m^2 - 84 < -3$
$\therefore$ discriminant is negative

$\Rightarrow$ no real roots

10. (a) **Use the binomial theorem** to express $(\sqrt{3}-\sqrt{2})^5$ in the form $a\sqrt{3}+b\sqrt{2}$, where $a$, $b$ are integers whose values are to be found. [5]

(b) Given that $(\sqrt{3}-\sqrt{2})^5 \approx 0$, use your answer to part (a) to find an approximate value for $\sqrt{6}$ in the form $\frac{c}{d}$, where $c$ and $d$ are positive integers whose values are to be found. [3]

a) $(\sqrt{3}-\sqrt{2})^5$ use pascal's triangle

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1

$(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$

$\therefore (\sqrt{3}-\sqrt{2})^5$

$= (\sqrt{3})^5 + 5(\sqrt{3})^4(-\sqrt{2}) + 10(\sqrt{3})^3(-\sqrt{2})^2 + 10(\sqrt{3})^2(-\sqrt{2})^3 + 5(\sqrt{3})(-\sqrt{2})^4 + (\sqrt{2})^5$

$= 9\sqrt{3} - 45\sqrt{2} + 60\sqrt{3} - 60\sqrt{2} + 20\sqrt{3} - 4\sqrt{2}$

$= 89\sqrt{3} - 109\sqrt{2}$

b) If $(\sqrt{3}-\sqrt{2})^5 \approx 0$ then $89\sqrt{3} - 109\sqrt{2} \approx 0$

$89\sqrt{3} \approx 109\sqrt{2}$ $(\times\sqrt{2})$

$89\sqrt{6} \approx 218$

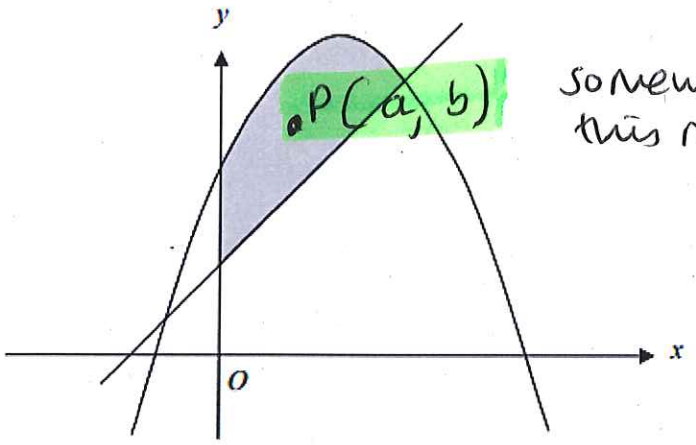
$\sqrt{6} \approx \frac{218}{89}$

or $89\sqrt{3} \approx 109\sqrt{2}$ $(\times\sqrt{3})$

$267 \approx 109\sqrt{6}$

$\frac{267}{109} \approx \sqrt{6}$

11.




somewhere in this region

The diagram shows a sketch of the curve $y = 6 + 4x - x^2$ and the line $y = x + 2$. The point $P$ has coordinates $(a, b)$. Write down the three inequalities involving $a$ and $b$ which are such that the point $P$ will be strictly contained within the shaded area above, if and only if, all three inequalities are satisfied. [3]

Think back to GCSE inequalities

Shaded region is

$x > 0$

$y > x + 2$

$y < 6 + 4x - x^2$

but this question specifies a point

$P = (a, b)$

so $x = a$ and $y = b$

substitute these into the inequalities

$a > 0$

$b > a + 2$

$b < 6 + 4a - a^2$

12. Prove that

$$\log_7 a \times \log_a 19 = \log_7 19$$

whatever the value of the positive constant $a$. [3]

Let $p = \log_a 19$

Rewrite as a power $a^p = 19$ (1)

Let $q = \log_7 a$

Rewrite as a power $7^q = a$ (2)

Substitute (2) into (1) gives

$(7^q)^p = 19$

$7^{qp} = 19$

Rewrite this as a logarithm

$\log_7(19) = qp$

but $p = \log_a 19$ and $q = \log_7 a$

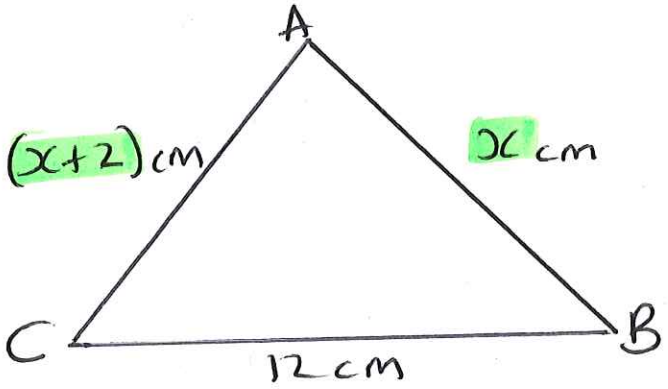
$\therefore \log_7 19 = \log_7 a \times \log_a 19$ as required

13. In triangle $ABC$, $BC = 12$ cm and $\cos A\hat{B}C = \frac{2}{3}$.
The length of $AC$ is 2 cm greater than the length of $AB$.

(a) Find the lengths of $AB$ and $AC$. [4]

(b) Find the exact value of $\sin B\hat{A}C$. Give your answer in its simplest form. [3]

a)




$\cos A\hat{B}C = \frac{2}{3}$

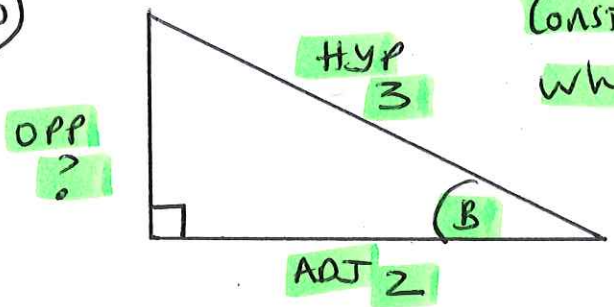
Using cosine rule $b^2 = a^2 + c^2 - 2ac\cos B$

$(x+2)^2 = 12^2 + x^2 - 2(12)(x)\left(\frac{2}{3}\right)$

$x^2 + 4x + 4 = 144 + x^2 - 16x$

$20x = 140$, $x = 7$, $AB = 7$cm, $AC = 9$cm

b)




Consider a right angled triangle where $\cos\hat{B} = \frac{2}{3}$

Using pythagoras

$(\text{OPPOSITE})^2 = 3^2 - 2^2$

$\text{OPPOSITE} = \sqrt{5}$

$\therefore \sin A\hat{B}C = \frac{\text{OPP}}{\text{HYP}} = \frac{\sqrt{5}}{3}$

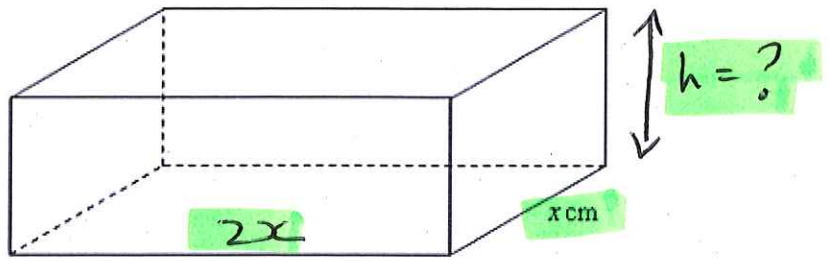
Now, back in original triangle

$\frac{\sin\hat{A}}{a} = \frac{\sin\hat{B}}{b}$

$\sin\hat{A} = \frac{\frac{\sqrt{5}}{3} \times 12}{9} = \frac{4\sqrt{5}}{9}$

$\sin B\hat{A}C = \frac{4\sqrt{5}}{9}$

14. The diagram below shows a closed box in the form of a cuboid, which is such that the length of its base is twice the width of its base. The volume of the box is 9000 cm$^3$. The total surface area of the box is denoted by $S$ cm$^2$.




(a) Show that $S = 4x^2 + \frac{27000}{x}$, where $x$ cm denotes the width of the base. [3]

(b) Find the minimum value of $S$, showing that the value you have found is a minimum value. [5]

a) $V = L \times W \times H$

$9000 = (2x) \times (x) \times H$

$\frac{9000}{2x^2} = H$

Total surface area = (2 × front) + (2 × top) + (2 × side)

$S = \left(2 \times 2x \times \frac{9000}{2x^2}\right) + \left(2 \times 2x \times x\right) + \left(2 \times x \times \frac{9000}{2x^2}\right)$

$S = \frac{18000}{x} + 4x^2 + \frac{9000}{x} = \frac{4x^2 + 27000}{x}$

b) $\frac{dS}{dx} = 8x - \frac{27000}{x^2}$

max/min values occur when $\frac{dS}{dx} = 0$

$\frac{27000}{x^2} = 8x$

$3375 = x^3$

$x = \sqrt[3]{3375}$

$x = 15$ cm

(Need to test to see if this is a max/min)

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$$\frac{d^2S}{dx^2} = 8 - (-2)27000x^{-3}$$

$$= 8 + \frac{54000}{x^3}$$

at $x = 15$, $\frac{d^2S}{dx^2} = 8 + \frac{54000}{(15)^3} > 0$

$\therefore$ Minimum value

$\therefore$ Minimum value of $S$

$$= 4(15)^2 + \frac{27000}{15}$$

$$= 2700 \text{ m}^2$$

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16. Find the range of values of $x$ for which the function

$$f(x) = x^3 - 5x^2 - 8x + 13$$

is an increasing function. [5]

$f(x) = x^3 - 5x^2 - 8x + 13$

$f'(x) = 3x^2 - 10x - 8$

$= 3x^2 + 2x - 12x - 8$

$= x(3x+2) - 4(3x+2)$

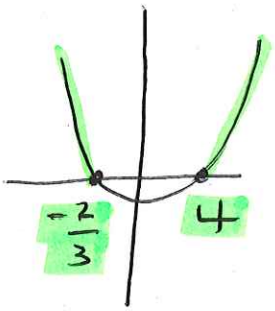
$= (3x+2)(x-4)$

| $x$ ac | $+b$ |
|---|---|
| $-24$ | $-10$ |
| | $-12, 2$ |

Then, the function is increasing when

$(3x+2)(x-4) > 0$

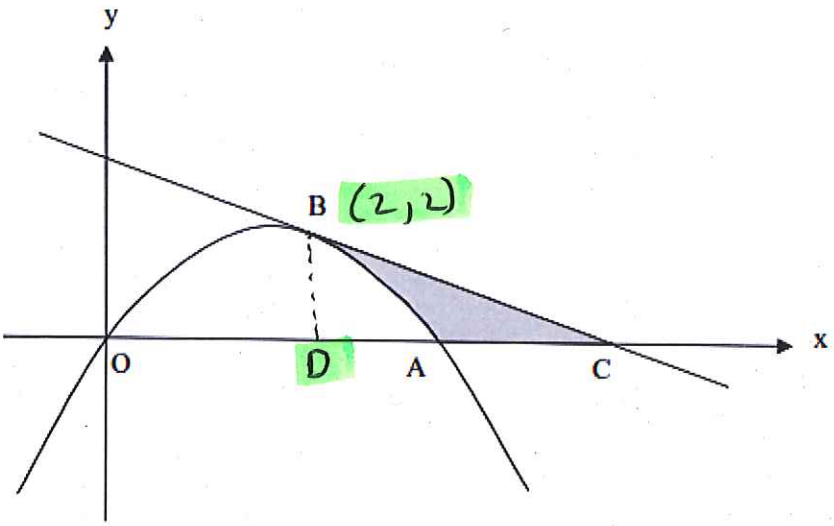
critical values are $x = -\frac{2}{3}$, $x = 4$




Function is increasing when

$x < -\frac{2}{3}$ or $x > 4$

17.




The diagram above shows a sketch of the curve $y = 3x - x^2$. The curve intersects the $x$-axis at the origin and at the point $A$. The tangent to the curve at the point $B(2, 2)$ intersects the $x$-axis at the point C.

(*a*) Find the equation of the tangent to the curve at *B*. [4]

(*b*) Find the area of the shaded region. [8]

a) $y = 3x - x^2$

$\frac{dy}{dx} = 3 - 2x$

at B (2,2) gradient of tangent $= 3 - 2(2) = -1$

Now, equation of tangent is given by $(y - y_1) = m(x - x_1)$ where $(x_1, y_1) = (2,2)$ and gradient $= -1$

$y - 2 = -1x - (-1)(2)$

$y = -x + 4$

b) Required Area = Area of Triangle DCB − Area under curve between D and A

MATHSDIY

To find area under curve we need the $x$ coordinates of A and D

$x$ coord of D = 2

A = where curve cuts $x$ axis

$y = 3x - x^2$

$y = x(3-x)$

cuts $x$ axis at $x=0, x=3$

$\therefore$ $x$ coord of A = 3

$x$ coord of C is where tangent

$y = -x+4$ cuts $x$ axis (so $y=0$)

$x$ coord of C = 4

Required Area = Area $\Delta$ − Area under curve

$$= \frac{2 \times 2}{2} - \int_2^3 (3x - x^2)\,dx$$

$$= 2 - \left[ \frac{3x^2}{2} - \frac{x^3}{3} \right]_2^3$$

$$= 2 - \left[ \left( \frac{27}{2} - 9 \right) - \left( 6 - \frac{8}{3} \right) \right]$$

$$= 2 - \frac{7}{6} = \frac{5}{6}$$

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18. (a) The vectors **u** and **v** are defined by $\mathbf{u} = 2\mathbf{i} - 3\mathbf{j}$, $\mathbf{v} = -4\mathbf{i} + 5\mathbf{j}$.

(i) Find the vector $4\mathbf{u} - 3\mathbf{v}$.

(ii) The vectors **u** and **v** are the position vectors of the points *U* and *V*, respectively. Find the length of the line *UV*. [4]

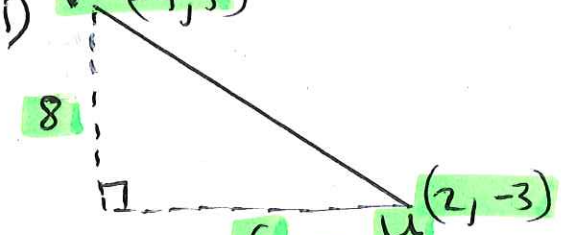
(b) Two villages *A* and *B* are 40 km apart on a long straight road passing through a desert. The position vectors of *A* and *B* are denoted by **a** and **b**, respectively.

(i) Village *C* lies on the road between *A* and *B* at a distance 4 km from *B*. Find the position vector of *C* in terms of **a** and **b**.

(ii) Village *D* has position vector $\frac{2}{9}\mathbf{a} + \frac{5}{9}\mathbf{b}$. Explain why village *D* cannot possibly be on the straight road passing through *A* and *B*. [3]

a) $4\underline{u} - 3\underline{v} = 4(2\underline{i} - 3\underline{j}) - 3(-4\underline{i} + 5\underline{j}) = \underline{20i - 27j}$

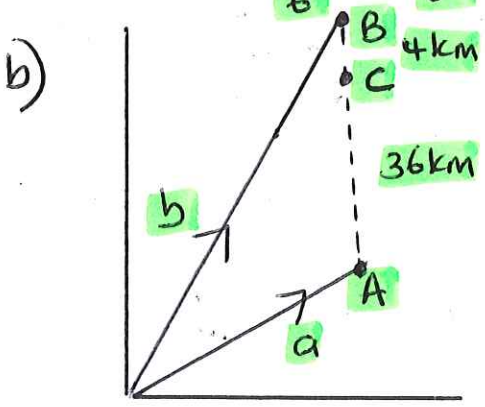
ii)




Magnitude of $UV = \sqrt{6^2 + 8^2}$

$= \sqrt{36 + 64} = 10$

b)




$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$= \underline{b} - \underline{a}$

$\overrightarrow{OC} = \overrightarrow{OA} + \frac{36}{40}(\overrightarrow{AB})$

$\overrightarrow{OC} = \underline{a} + \frac{9}{10}(\underline{b} - \underline{a})$

$\overrightarrow{OC} = \frac{1}{10}\underline{a} + \frac{9}{10}\underline{b}$

ii) Any village on AB will have a position vector of the form $\underline{a} + \lambda(\underline{b} - \underline{a})$

(like village C above but we don't know how far along $\overrightarrow{AB}$ it would be) $= a + \lambda\underline{b} - \lambda\underline{a}$

$= (1 - \lambda)\underline{a} + \lambda\underline{b}$

and $\frac{2}{9}\underline{a} + \frac{5}{9}\underline{b}$ does not satisfy this.

15. The size $N$ of the population of a small island at time $t$ years may be modelled by $N = Ae^{kt}$, where $A$ and $k$ are constants. It is known that $N = 100$ when $t = 2$ and that $N = 160$ when $t = 12$.

(a) Interpret the constant $A$ in the context of the question. [1]

(b) Show that $k = 0{\cdot}047$, correct to three decimal places. [4]

(c) Find the size of the population when $t = 20$. [3]

a) A would represent the initial population because $N = Ae^{kt}$ and $t=0$ gives $N=A$.

b) $N=100$ when $t=2$ gives $100 = Ae^{2k}$ ①

$N=160$ when $t=12$ gives $160 = Ae^{12k}$ ②

② ÷ ① gives $\frac{160}{100} = \frac{Ae^{12k}}{Ae^{2k}}$, $1{\cdot}6 = e^{10k}$

Rewrite as a logarithm

$\lg_e 1{\cdot}6 = 10k$

$0{\cdot}47000\ldots = 10k$

$k = 0{\cdot}047$ (3dp)

c) $N = Ae^{0.047t}$

Find A in ① $100 = Ae^{2(0.047)}$

$A = 91{\cdot}03$ (2dp)

$N = 91{\cdot}03e^{0.047t}$

when $t=20$

$N = 91{\cdot}03e^{0.047\times 20}$

$N = 233$