



GCE AS/A Level MATHEMATICS

**GCE AS/A Level FURTHER
MATHEMATICS**

FORMULA BOOKLET

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Pure Mathematics

Mensuration

$$\text{Surface area of sphere} = 4\pi r^2$$

$$\text{Area of curved surface of cone} = \pi r \times \text{slant height}$$

Arithmetic Series

$$S_n = \frac{1}{2}n(a+l) = \frac{1}{2}n[2a+(n-1)d]$$

Geometric Series

$$S_n = \frac{a(1-r^n)}{1-r}$$

$$S_\infty = \frac{a}{1-r} \quad \text{for } |r| < 1$$

Summations

$$\sum_{r=1}^n r^2 = \frac{1}{6}n(n+1)(2n+1)$$

$$\sum_{r=1}^n r^3 = \frac{1}{4}n^2(n+1)^2$$

Binomial Series

$$\binom{n}{r} + \binom{n}{r+1} = \binom{n+1}{r+1}$$

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \dots + \binom{n}{r}a^{n-r}b^r + \dots + b^n \quad (n \in \mathbb{N})$$

$$\text{where } \binom{n}{r} = {}^nC_r = \frac{n!}{r!(n-r)!}$$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{1 \cdot 2}x^2 + \dots + \frac{n(n-1)\dots(n-r+1)}{1 \cdot 2 \dots r}x^r + \dots \quad (|x| < 1, n \in \mathbb{R})$$

Logarithms and exponentials

$$e^{x \ln a} = a^x$$

Complex Numbers

$$\{r(\cos \theta + i \sin \theta)\}^n = r^n(\cos n\theta + i \sin n\theta)$$

The roots of $z^n = 1$ are given by $z = e^{\frac{2\pi ki}{n}}$, for $k = 0, 1, 2, \dots, n-1$

Maclaurin's and Taylor's Series

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \dots + \frac{x^r}{r!} f^{(r)}(0) + \dots$$

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!} f''(a) + \dots + \frac{(x-a)^r}{r!} f^{(r)}(a) + \dots$$

$$f(a+x) = f(a) + xf'(a) + \frac{x^2}{2!} f''(a) + \dots + \frac{x^r}{r!} f^{(r)}(a) + \dots$$

$$e^x = \exp(x) = 1 + x + \frac{x^2}{2!} + \dots + \frac{x^r}{r!} + \dots \quad \text{for all } x$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots + (-1)^{r+1} \frac{x^r}{r} + \dots \quad (-1 < x \leq 1)$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots + (-1)^r \frac{x^{2r+1}}{(2r+1)!} + \dots \quad \text{for all } x$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots + (-1)^r \frac{x^{2r}}{(2r)!} + \dots \quad \text{for all } x$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots + (-1)^r \frac{x^{2r+1}}{2r+1} + \dots \quad (-1 \leq x \leq 1)$$

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots + \frac{x^{2r+1}}{(2r+1)!} + \dots \quad \text{for all } x$$

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots + \frac{x^{2r}}{(2r)!} + \dots \quad \text{for all } x$$

$$\tanh^{-1} x = x + \frac{x^3}{3} + \frac{x^5}{5} + \dots + \frac{x^{2r+1}}{2r+1} + \dots \quad (-1 < x < 1)$$

Hyperbolic Functions

$$\cosh^2 x - \sinh^2 x = 1$$

$$\sinh 2x = 2 \sinh x \cosh x$$

$$\cosh 2x = \cosh^2 x + \sinh^2 x$$

$$\cosh^{-1} x = \ln \left\{ x + \sqrt{x^2 - 1} \right\} \quad (x \geq 1)$$

$$\sinh^{-1} x = \ln \left\{ x + \sqrt{x^2 + 1} \right\}$$

$$\tanh^{-1} x = \frac{1}{2} \ln \left(\frac{1+x}{1-x} \right) \quad (|x| < 1)$$

Trigonometric Identities

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B} \quad \left(A \pm B \neq \left(k + \frac{1}{2}\right)\pi\right)$$

$$\text{For } t = \tan \frac{1}{2}A: \sin A = \frac{2t}{1+t^2}, \cos A = \frac{1-t^2}{1+t^2}$$

$$\sin A + \sin B = 2 \sin \frac{A+B}{2} \cos \frac{A-B}{2}$$

$$\sin A - \sin B = 2 \cos \frac{A+B}{2} \sin \frac{A-B}{2}$$

$$\cos A + \cos B = 2 \cos \frac{A+B}{2} \cos \frac{A-B}{2}$$

$$\cos A - \cos B = -2 \sin \frac{A+B}{2} \sin \frac{A-B}{2}$$

Vectors

The resolute of \mathbf{a} in the direction of \mathbf{b} is $\frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|}$.

The point dividing AB in the ratio $\lambda : \mu$ is $\frac{\mu \mathbf{a} + \lambda \mathbf{b}}{\lambda + \mu}$.

The equation of a plane in Cartesian form is $n_1x + n_2y + n_3z = k$.

The perpendicular distance between two skew lines is $D = \frac{|(\mathbf{b} - \mathbf{a}) \cdot \mathbf{n}|}{|\mathbf{n}|}$, where \mathbf{a} and \mathbf{b} are position vectors of points on each line and \mathbf{n} is a mutual perpendicular to both lines.

The perpendicular distance between a point and a line is $D = \frac{|ax_1 + by_1 - c|}{\sqrt{a^2 + b^2}}$, where the coordinates of the point are (x_1, y_1) and the equation of the line is given by $ax + by = c$.

The perpendicular distance between a point and a plane is $D = \frac{|n_1\alpha + n_2\beta + n_3\gamma - k|}{\sqrt{n_1^2 + n_2^2 + n_3^2}}$, where

(α, β, γ) are the coordinates of the point and $n_1x + n_2y + n_3z = k$ is the equation of the plane.

Matrix transformations in 2-D

Anticlockwise rotation through θ about O : $\begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$

Reflection in the line $y = (\tan \theta)x$: $\begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$

Matrix transformations in 3-D

Anticlockwise rotation through θ about:

x-axis	y-axis	z-axis
$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta \\ 0 & \sin \theta & \cos \theta \end{bmatrix}$	$\begin{bmatrix} \cos \theta & 0 & \sin \theta \\ 0 & 1 & 0 \\ -\sin \theta & 0 & \cos \theta \end{bmatrix}$	$\begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

where

- an anticlockwise (or positive) rotation about Ox (or x -axis) is in the sense $\mathbf{j} \rightarrow \mathbf{k}$,
- an anticlockwise (or positive) rotation about Oy (or y -axis) is in the sense $\mathbf{k} \rightarrow \mathbf{i}$,
- an anticlockwise (or positive) rotation about Oz (or z -axis) is in the sense $\mathbf{i} \rightarrow \mathbf{j}$.

Differentiation

Function	Derivative
$\frac{f(x)}{g(x)}$	$\frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$
$\tan x$	$\sec^2 x$
$\sec x$	$\sec x \tan x$
$\cot x$	$-\operatorname{cosec}^2 x$
$\operatorname{cosec} x$	$-\operatorname{cosec} x \cot x$
$\sin^{-1} x$	$\frac{1}{\sqrt{1-x^2}}$
$\cos^{-1} x$	$-\frac{1}{\sqrt{1-x^2}}$
$\tan^{-1} x$	$\frac{1}{1+x^2}$
$\sinh x$	$\cosh x$
$\cosh x$	$\sinh x$
$\tanh x$	$\operatorname{sech}^2 x$
$\sinh^{-1} x$	$\frac{1}{\sqrt{1+x^2}}$
$\cosh^{-1} x$	$\frac{1}{\sqrt{x^2-1}}$
$\tanh^{-1} x$	$\frac{1}{1-x^2}$

Integration (+ constant; $a > 0$ where relevant)

Function	Integral
$\tan x$	$\ln \sec x $
$\cot x$	$\ln \sin x $
$\operatorname{cosec} x$	$-\ln \operatorname{cosec} x + \cot x = \ln \left \tan \left(\frac{1}{2} x \right) \right $
$\sec x$	$\ln \sec x + \tan x = \ln \left \tan \left(\frac{1}{2} x + \frac{1}{4} \pi \right) \right $
$\sec^2 x$	$\tan x$
$\sinh x$	$\cosh x$
$\cosh x$	$\sinh x$
$\tanh x$	$\ln \cosh x$
$\frac{1}{\sqrt{a^2 - x^2}}$	$\sin^{-1} \left(\frac{x}{a} \right)$ ($ x < a$)
$\frac{1}{a^2 + x^2}$	$\frac{1}{a} \tan^{-1} \left(\frac{x}{a} \right)$
$\frac{1}{\sqrt{x^2 - a^2}}$	$\cosh^{-1} \left(\frac{x}{a} \right) = \ln \left\{ x + \sqrt{x^2 - a^2} \right\}$ ($x > a$)
$\frac{1}{\sqrt{a^2 + x^2}}$	$\sinh^{-1} \left(\frac{x}{a} \right) = \ln \left\{ x + \sqrt{x^2 + a^2} \right\}$
$\frac{1}{a^2 - x^2}$	$\frac{1}{2a} \ln \left \frac{a+x}{a-x} \right = \frac{1}{a} \tanh^{-1} \left(\frac{x}{a} \right)$ ($ x < a$)
$\frac{1}{x^2 - a^2}$	$\frac{1}{2a} \ln \left \frac{x-a}{x+a} \right $

$$\int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx$$

Area of a sector

$$A = \frac{1}{2} \int r^2 d\theta \quad (\text{polar coordinates})$$

Numerical Mathematics

Numerical integration

The trapezium rule: $\int_a^b y \, dx \approx \frac{1}{2} h \{(y_0 + y_n) + 2(y_1 + y_2 + \dots + y_{n-1})\}$, where $h = \frac{b-a}{n}$

Numerical Solution of Equations

The Newton-Raphson iteration for solving $f(x) = 0$: $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

Mechanics

Motion in a circle

Transverse velocity: $v = r\dot{\theta} = \omega r$

Radial acceleration: $-r\dot{\theta}^2 = -\frac{v^2}{r} = -\omega^2 r$

Centres of Mass of Uniform Bodies

Triangular lamina: $\frac{2}{3}$ along median from vertex

Semi circle: $\frac{4r}{3\pi}$ from straight edge along axis of symmetry

Quarter circle: $\bar{x} = \frac{4r}{3\pi}$ $\bar{y} = \frac{4r}{3\pi}$ from vertex

Probability & Statistics

Probability

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$P(A \cap B) = P(A)P(B | A)$$

$$P(A | B) = \frac{P(A)P(B | A)}{P(A)P(B | A) + P(A')P(B | A')}$$

$$\text{Bayes' Theorem: } P(A_j | B) = \frac{P(A_j)P(B | A_j)}{\sum P(A_i) P(B | A_i)}$$

Discrete distributions

For a discrete random variable X taking values x_i with probabilities p_i

$$\text{Expectation (mean): } E(X) = \mu = \sum x_i p_i$$

$$\text{Variance: } \text{Var}(X) = \sigma^2 = \sum (x_i - \mu)^2 p_i = \sum x_i^2 p_i - \mu^2$$

$$\text{For a function } g(X): E(g(X)) = \sum g(x_i) p_i$$

Standard discrete distributions:

Distribution of X	$P(X = x)$	Mean	Variance
Binomial $B(n, p)$	$\binom{n}{x} p^x (1-p)^{n-x}$	np	$np(1-p)$
Poisson $Po(\lambda)$	$e^{-\lambda} \frac{\lambda^x}{x!}$	λ	λ

Continuous distributions

For a continuous random variable X having probability density function f

$$\text{Expectation (mean): } E(X) = \mu = \int x f(x) dx$$

$$\text{Variance: } \text{Var}(X) = \sigma^2 = \int (x - \mu)^2 f(x) dx = \int x^2 f(x) dx - \mu^2$$

$$\text{For a function } g(X): E(g(X)) = \int g(x) f(x) dx$$

$$\text{Cumulative distribution function: } F(x) = P(X \leq x) = \int_{-\infty}^x f(t) dt$$

Standard continuous distributions:

Distribution of X	P.D.F.	Mean	Variance
Uniform (Rectangular) on $[a, b]$ $U[a, b]$	$\frac{1}{b-a}$	$\frac{1}{2}(a+b)$	$\frac{1}{12}(b-a)^2$
Normal $N(\mu, \sigma^2)$	$\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$	μ	σ^2
Exponential $\text{Exp}(\lambda)$	$\lambda e^{-\lambda x}$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$

Expectation algebra

For independent random variables X and Y

$$E(XY) = E(X)E(Y), \quad \text{Var}(aX \pm bY) = a^2\text{Var}(X) + b^2\text{Var}(Y)$$

Sampling distributions

For a random sample X_1, X_2, \dots, X_n of n independent observations from a distribution having mean μ and variance σ^2

$$\bar{X} \text{ is an unbiased estimator of } \mu, \text{ with } \text{Var}(\bar{X}) = \frac{\sigma^2}{n}$$

$$S^2 \text{ is an unbiased estimator of } \sigma^2, \text{ where } S^2 = \frac{\sum (X_i - \bar{X})^2}{n-1}$$

For a random sample of n observations from $N(\mu, \sigma^2)$

$$\frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \sim N(0,1)$$

$$\frac{\bar{X} - \mu}{S/\sqrt{n}} \sim t_{(n-1)}$$

If X is the observed number of successes in n independent Bernoulli trials in each of which the probability of success is p , and $Y = \frac{X}{n}$, then

$$E(Y) = p \quad \text{and} \quad \text{Var}(Y) = \frac{p(1-p)}{n}$$

For a random sample of n_x observations from $N(\mu_x, \sigma_x^2)$ and, independently, a random sample of n_y observations from $N(\mu_y, \sigma_y^2)$

$$\frac{(\bar{X} - \bar{Y}) - (\mu_x - \mu_y)}{\sqrt{\frac{\sigma_x^2}{n_x} + \frac{\sigma_y^2}{n_y}}} \sim N(0,1)$$

Correlation and Regression

For a sample of n pairs of observations (x_i, y_i)

$$S_{xx} = \sum (x_i - \bar{x})^2 = \sum x_i^2 - \frac{(\sum x_i)^2}{n}$$

$$S_{yy} = \sum (y_i - \bar{y})^2 = \sum y_i^2 - \frac{(\sum y_i)^2}{n}$$

$$S_{xy} = \sum (x_i - \bar{x})(y_i - \bar{y}) = \sum x_i y_i - \frac{(\sum x_i)(\sum y_i)}{n}$$

A measure of linear association between two variables X and Y is given by the Pearson product - moment correlation coefficient r .

For the sample $(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$, it is given by $r = \frac{S_{xy}}{\sqrt{S_{xx}S_{yy}}}$.

Given data, the parameters α and β of the linear regression model may be estimated using the principle of least squares.

The least squares estimate $\hat{\beta}$ of the parameter β is given by $\hat{\beta} = \frac{S_{xy}}{S_{xx}}$.

The least squares estimate $\hat{\alpha}$ of the parameter α is given by $\hat{\alpha} = \bar{y} - \hat{\beta}\bar{x}$.

The least squares regression line is given by $y = \hat{\alpha} + \hat{\beta}x$.

Spearman's rank correlation coefficient is given by $r_s = 1 - \frac{6\sum d_i^2}{n(n^2 - 1)}$.