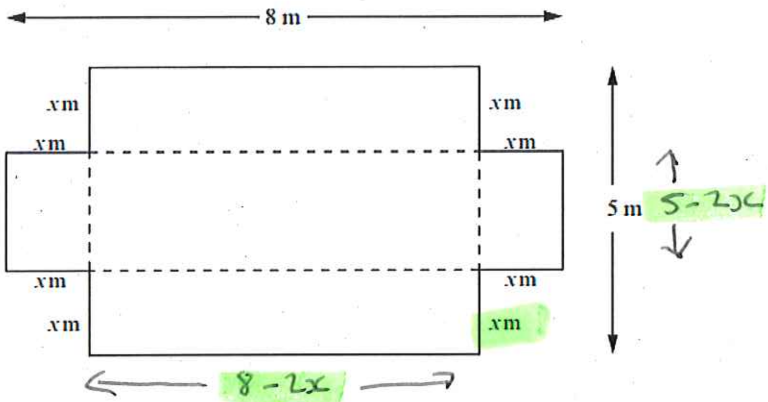
APPLICATIONS OF DIFFERENTIATION: OPTIMISATION

AS Unit 1: Pure Mathematics A

WJEC past paper questions: 2010 – 2017

Total marks available 37 (approximately 45 minutes)

1. A rectangular sheet of metal has length 8m and width 5m. Four squares, each of side xm, where $x < 2.5$, have been cut away from the corners of the rectangular sheet, as shown in the diagram below. The rest of the metal sheet is now bent along the dotted lines to form an open tank in the form of a cuboid.



a) Show that the volume Vm^3 of this tank is given by
$$V = 4x^3 - 26x^2 + 40x\,. \qquad (2)$$

b) Find the maximum value of V, showing that the value you have found is a maximum value. (5)

(May 11)

1a) Volume of rectangular tank = L × W × H

$= (8-2x)(5-2x)(x)$

$= (40 - 10x - 16x + 4x^2)(x) = 4x^3 - 26x^2 + 40x$

b) $\frac{dV}{dx} = 12x^2 - 52x + 40$, stationary values occur at $\frac{dV}{dx} = 0$

$12x^2 - 52x + 40 = 0 \ (\div 4)$

$3x^2 - 13x + 10 = 0$

$(3x - 10)(x - 1) = 0 \qquad x = 3\tfrac{1}{3}$ (not possible) or $x = 1$

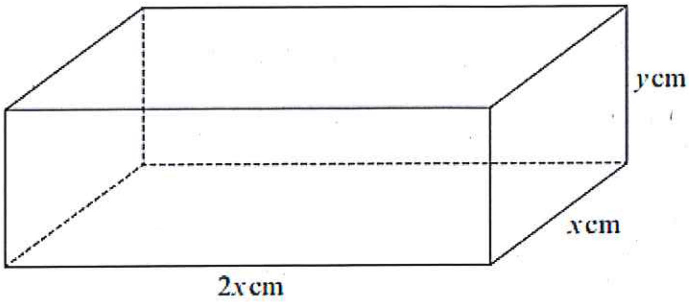
$\frac{d^2V}{dx^2} = 24x - 52$, at $x = 1$, $\frac{d^2V}{dx^2} < 0 \therefore$ maximum at $x = 1$

$8 - 2x = 8 - 2 = 6\text{m} =$ length

$5 - 2x = 5 - 2 = 3\text{m} =$ width

$\therefore$ Volume $= 6 \times 3 \times 1 = 18\text{m}^3$ which is the maximum

2. The diagram shows a **closed** box in the form of a cuboid. The length of the box is $2x$ cm, its width is x cm and its height is y cm.



The total surface area of the box is 108cm^2.

a) i) Write down an equation involving x and y and hence show that
$$xy = 18 - \frac{2}{3}x^2 .$$
ii) Hence show that the volume Vcm^3 of the box is given by
$$V = 36x - \frac{4}{3}x^3 .$$ (3)

b) Find the maximum value of V, showing that the value you have found is a maximum value. (5)

(Summer 13)

2a i) Total surface area = 2× (front + side + top)
$= 2\times(2xy + xy + 2x^2) = 6xy + 4x^2$
Given that surface area = 108cm^2
$\therefore \quad 6xy + 4x^2 = 108$
$6xy = 108 - 4x^2 \quad (\div 6)$
$xy = 18 - \frac{2}{3}x^2$ as required

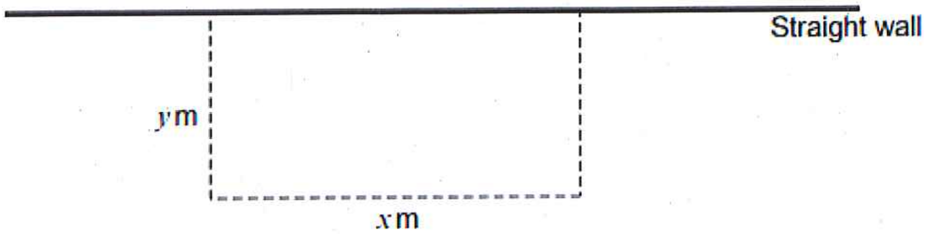
ii) Volume of box = L × W × H = $2x \times x \times y = 2x \times xy$
$= 2x\left(18 - \frac{2}{3}x^2\right)$
$= 36x - \frac{4}{3}x^3$

b) $\frac{dV}{dx} = 36 - 3\left(\frac{4}{3}\right)x^2 = 36 - 4x^2$
Stationary values occur when $\frac{dV}{dx} = 0$,
$36 - 4x^2 = 0$
$36 = 4x^2$
$9 = x^2, \; x = \pm 3$
x cannot be negative.
$x = 3$cm

Is this a maximum?
$\frac{d^2V}{dx^2} = -8x$
at $x = 3$, $\frac{d^2V}{dx^2} < 0 \; \therefore$ maximum

Maximum volume $= 36(3) - \frac{4}{3}(3)^3 = 108 - 36$
$= 72$cm^3

3. A sheep farmer wishes to construct a rectangular enclosure for his animals. He decides to use a straight wall as one side of the enclosure and fencing for the other three sides. The area of the enclosure is to be 800m^2. The lengths of the sides of the rectangular enclosure are x m and y m, as shown in the diagram, and the total length of the **fencing** is Lm.



a) Show that $L = x + \frac{1600}{x}$ (2)

b) Find the minimum value of L, showing that the value you have found is a minimum value. (5)

(Summer 15)

3a) Area $= 800\text{m}^2$ $\therefore$ $L \times W = 800$

$xy = 800$, $y = \frac{800}{x}$

Total length of fencing $L = x + 2y$, but $y = \frac{800}{x}$

$L = x + \frac{1600}{x}$ as required.

$(= x + 1600x^{-1})$

b) $\frac{dL}{dx} = 1 - 1600x^{-2}$

$= 1 - \frac{1600}{x^2}$, stationary values when $\frac{dL}{dx} = 0$

$1 - \frac{1600}{x^2} = 0$

$1 = \frac{1600}{x^2}$

$x^2 = 1600$

$x = \pm 40$, cannot have negative length, $x = 40\text{m}$

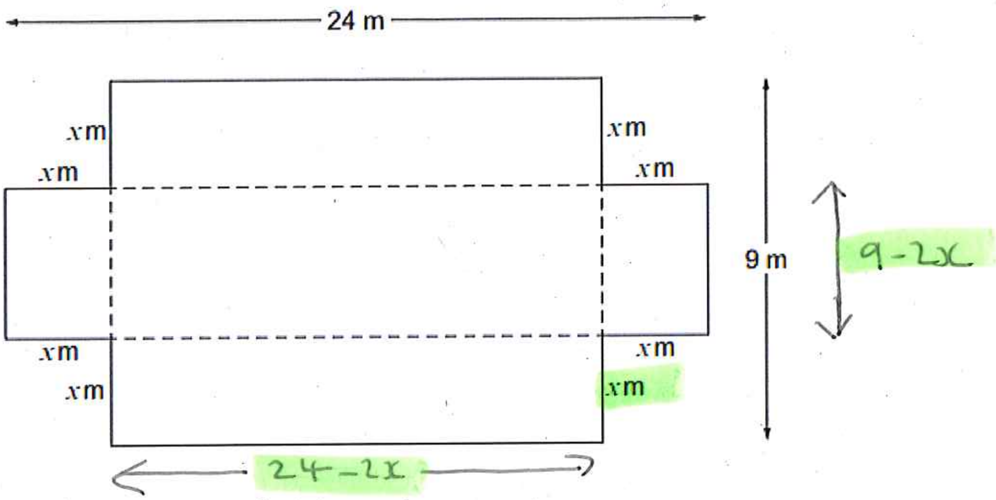
is this a minimum?

$\frac{d^2L}{dx^2} = 3200x^{-3}$, at $x = 40$ $\frac{d^2L}{dx^2} > 0$ $\therefore$ minimum

$L = x + \frac{1600}{x}$, at $x = 40$ which is a minimum

$L = 40 + \frac{1600}{40} = 40 + 40 = 80\text{m}$

4. A rectangular sheet of metal has length 24m and width 9m. Four squares, each of side xm, where $x < 4.5$, have been cut away from the corners of the rectangular sheet, as shown in the diagram below. The rest of the metal sheet is now bent along the dotted lines to form an open tank in the form of a cuboid.



a) Show that the volume $V\,\text{m}^3$ of this tank is given by

$$V = 4x^3 - 66x^2 + 216x$$ (2)

b) Find the maximum value of V, showing that the value you have found is a maximum value. (5)

(Summer 16)

4 a) Volume of rectangular tank = L x W x H

$= (24-2x)(9-2x)(x)$

$= (216 - 18x - 48x + 4x^2)x = 4x^3 - 66x^2 + 216x$

b) $\frac{dV}{dx} = 12x^2 - 132x + 216$, stationary values occur at $\frac{dV}{dx} = 0$

$12x^2 - 132x + 216 = 0 \quad (\div 12)$

$x^2 - 11x + 18 = 0$

$(x-9)(x-2) = 0$, two possibilities $x = 9m$, $x = 2m$

but, width is $9-2x$, so '9m' won't work. $\therefore x = 2m$

Is this a maximum?

$\frac{d^2V}{dx^2} = 24x - 132$, at $x = 2$ $\frac{d^2V}{dx^2} = 24(2) - 132 < 0$

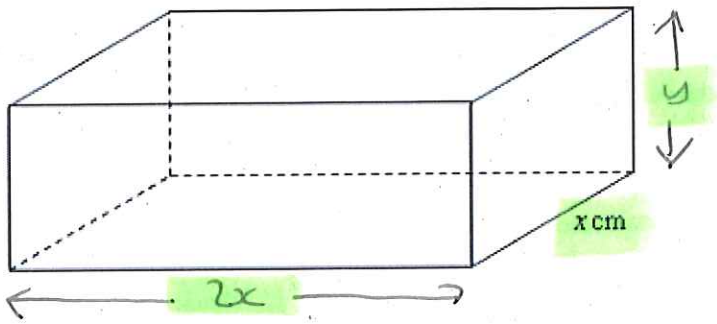
$\therefore$ MAXIMUM

Maximum volume $= 4(2)^3 - 66(2)^2 + 216(2)$

$= 32 - 264 + 432$

$= 200m^3$

5. The diagram below shows a closed box in the form of a cuboid, which is such that the length of its base it twice the width of its base. The volume of the box is 9000cm^3. The total surface area of the box is denoted by Scm^2.



(a) Show that $S = 4x^2 + \frac{27000}{x}$, where $x$cm denotes the width of the base. (3)

(b) Find the minimum value of S, showing that the value you have found is a minimum value. (5)

(Sample Paper)

5a) Total surface area = 2× (front + side + top)

and length = 2x, width = x, height = y

∴ Total surface area = $2\times(2xy + xy + 2x^2) = 6xy + 4x^2$ (1)

Given, volume = 9000, therefore $2x \times x \times y = 9000$

$y = \frac{9000}{2x^2}$

$y = \frac{4500}{x^2}$ (2)

(2) in (1) gives $S = 6x\left(\frac{4500}{x^2}\right) + 4x^2$

$S = 4x^2 + \frac{27000}{x}$ $= 4x^2 + 27000x^{-1}$

b) $\frac{dS}{dx} = 8x - 27000x^{-2}$

stationary values when $\frac{dS}{dx} = 0$, $8x = \frac{27000}{x^2}$

$x^3 = \frac{27000}{8}$, $x = 15$cm

is this a minimum?

$\frac{d^2S}{dx^2} = 8 + 54000x^{-3}$, at $x = 15$, $\frac{d^2S}{dx^2} = 8 + \frac{54000}{15^3} > 0$ ∴ MINIMUM

at $x = 15$, $S = 4(15)^2 + \frac{27000}{15} = 2700$m^2 & this is a MINIMUM value