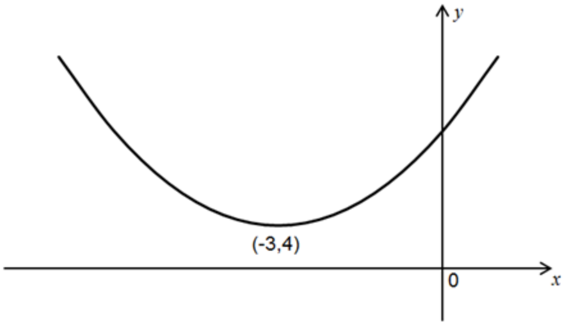
A2 Mathematics Unit 3: Pure Mathematics B

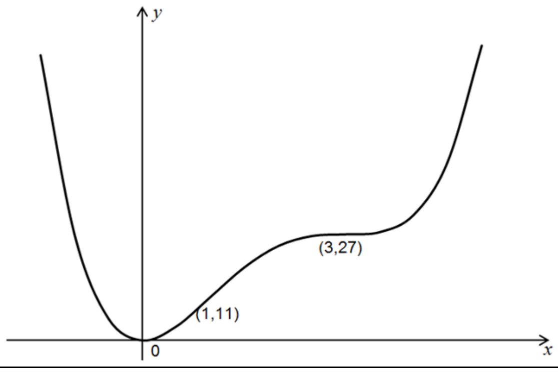
Solutions and Mark Scheme

Question Number	Solution	Mark	AO	Notes
1. (a)	$1-\frac{x^2}{2}-4x=x^2$	M1	AO1	(Attempt to substitute for $\cos x, \sin x$)
	$\frac{3x^2}{2}+4x-1=0$	A1	AO1	(Correct)
	$3x^2+8x-2=0$	B1	AO1	
	$x=\frac{-8\pm\sqrt{64+24}}{6}=\frac{-8\pm\sqrt{88}}{6}$			
	$x=0.230(1385...)$, $(-2.896805...)$	B1	AO1	
		[4]		
2.	$V=\frac{4}{3}\pi r^3$			
	$\frac{dV}{dt}=3\times\frac{4}{3}\pi r^2\frac{dr}{dt}$	B1	AO3	
	$4\pi\times 15^2\frac{dr}{dt}=250$	M1	AO3	(Substitution of data)
	$\frac{dr}{dt}=\frac{250}{900\pi}\approx 0.088$ (cm/second)	A1	AO3	
		[3]		

Question Number	Solution	Mark	AO	Notes
3. (a)		G1 G1	AO1 AO1	(Shape) (Stationary point)
(b) (i)	A correct statement, eg. f^{-1} doesn't exist because f is not a one-one function	E1	AO2	
(ii)	Any appropriate domain eg. There are many possible appropriate domains. It is essential that any domain must be contained in one branch of the curve shown.	B1	AO2	
	Here we consider $(-3, \infty)$. Let $y = x^2 + 6x + 13$ $= (x+3)^2 + 4$ $x + 3 = \pm\sqrt{y-4}$	M1	AO1	(Attempt to find x in terms of y)
	So that $x = -3 \pm \sqrt{y-4}$	A1	AO1	
	Since $x > -3$, the positive sign is appropriate $\therefore x = -3 + \sqrt{y-4}$	A1	AO2	
	And $f^{-1}(x) = -3 + \sqrt{x-4}$	A1	AO2	
		[8]		

Question Number	Solution	Mark	AO	Notes
4. (a)	$(1-x)^{-\frac{1}{2}} = 1 + \frac{x}{2} + \left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\frac{x^2}{2} + ..$			
	$= 1 + \frac{x}{2} + \frac{3x^2}{8} + ...$	B1	AO1	
	Valid for $\lvert x \rvert < 1$	B1	AO1	
	When $x = \frac{1}{10}$, $\left(\frac{9}{10}\right)^{-\frac{1}{2}} \approx 1 + \frac{1}{20} + \frac{3}{800} = \frac{843}{800}$	B1	AO2	
	So that $(10)^{\frac{1}{2}} = 3\text{x}\frac{843}{800} = \frac{2529}{800}$	B1	AO1	
		[4]		
5.	After 30 years, saving is			
	$(1.08)1000 + (1.08)^2 1000 + ... + (1.08)^{30} 1000$	B1	AO3	
	This is G.P with $a = (1 \cdot 08)1000$			
	$r = 1 \cdot 08$			
	and $n = 30$	B2	AO3,AO3	(B2 for 3 correct, B1 for 2 correct)
	Then			
	$S_{30} = (1000)(1.08)\left(\frac{(1.08)^{30} - 1}{0.08}\right)$	M1	AO3	(correct formula)
	$\approx$ £122,346	A1	AO3	
		[5]		

Question Number	Solution	Mark	AO	Notes
6.	If smallest side is a, largest side $=8a$			
	$8a = a + 14d$	M1	AO3	(Attempt to relate the two sides)
	$a = 2d$	A1	AO3	
	Perimeter $=\frac{15}{2}\left[2a+14d\right]=\frac{15}{2}.18d=135d$	M1	AO3	
	$\therefore 135d = 270$			
	$d = 2$			
	Length of smallest side $= a = 2d = 4$ cm	B1	AO3	
	Alternative mark scheme:			
	smallest side = a, largest side $=8a$			
	Perimeter $=\frac{15}{2}\left[a+8a\right]=\frac{15}{2}.9a=\frac{135}{2}a$	(M1)	(AO3)	
		(A1)	(AO3)	
	$\therefore \frac{135}{2}a = 270$	(M1)	(AO3)	
	$a = 4$	(A1)	(AO3)	
	Length of smallest side $= a = 4$ cm	**[4]**		

Question Number	Solution	Mark	AO	Notes
7. (a)	$\frac{d^2y}{dx^2} = 12ax^2 + 6bx + 36$	M1	AO2	(attempt to find $\frac{d^2y}{dx^2}$, 2 correct terms)
	For point of inflection at $(1,11)$ $12a + 6b + 36 = 0$ So that $2a + b + 6 = 0$ (1)	A1	AO2	
(b)	Also $a + b + 18 = 11$ (2)	B1 M1	AO1 AO1	(Attempt to solve for *a*, *b*)
	From (1), (2), $a = 1, \quad b = -8$	A1	AO1	
	$\therefore \frac{d^2y}{dx^2} = 12x^2 - 48x + 36$ $= 12(x^2 - 4x + 3) = 12(x-1)(x-3) = 0$	M1	AO2	
	$\therefore \frac{d^2y}{dx^2} = 0$ when $x = 3$	A1	AO2	
	and $\frac{d^2y}{dx^2}$ changes sign as *x* passes through 3	m1	AO2	
	$\therefore$ There is a point of inflection at $x = 3, \quad y = 3^4 - 8.3^3 + 18.3^2 = 27$, i.e at $(3, 27)$	A1 A1	AO2 AO2	(Only if m1 is awarded)
(c)	$\frac{dy}{dx} = 4x^3 - 24x^2 + 36x = 0$	M2	AO1,AO1	(M1 for correct differentiation but not equal to 0)
	$\therefore 4x(x^2 - 6x + 9) = 0$ giving $x = 0, x = 3$	A1	AO1	(point of Inflection) (Two Values)
	Then at $x = 0, y = 0$ and $\frac{d^2y}{dx^2} = 36$ There is a minimum at $x = 0, y = 0$	A1	AO1	
		G1	AO1	general shape
		G1	AO1	min two points of inflection
		[16]		

Question Number	Solution	Mark	AO	Notes
8 (a) (i)	$-\frac{e^{-3x+5}}{3}+C$	M1 A1	AO1 AO1	($ke^{-3x+5)}$) ($k=-\frac{1}{3}$)
(ii)	$\int x^2 \ln x \, dx$			
	$u=\ln x, \frac{dv}{dx}=x^2$	M1	AO1	(Correct u and $\frac{dv}{dx}$)
	$\frac{du}{dx}=\frac{1}{x}, \; v=\frac{x^3}{3}$			
	$\int x^2 \ln x \, dx = \frac{x^3}{3}\ln x - \int \frac{x^3}{3}.\frac{1}{x}dx$	A1,A1	AO1, AO1	
	$= \frac{x^3}{3}\ln x - \frac{x^3}{9}+C$	A1	AO1	
	(Penalise omission of C once only)			
(b)	$\int_0^{\frac{1}{2}} \frac{x^2}{\sqrt{1-x^2}} dx$			
	$x=\sin\theta \qquad dx=\cos\theta \, d\theta$	B1	AO3	
	$x=0, \; \theta=0 \qquad x=\frac{1}{2}, \; \theta=\frac{\pi}{6}$	B1	AO3	
	$=\int_0^{\frac{\pi}{6}} \frac{\sin^2\theta\cos\theta}{\sqrt{1-\sin^2\theta}} d\theta$	M1	AO3	(attempt to substitute)
	$=\int_0^{\frac{\pi}{6}} \frac{\sin^2\theta}{\cos\theta}\cos\theta \, d\theta$	A1	AO3	(Correct)
	$=\int_0^{\frac{\pi}{6}} \sin^2\theta \, d\theta$	A1	AO3	
	$=\int_0^{\frac{\pi}{6}} \frac{1-\cos 2\theta}{2} d\theta$	m1	AO3	
	$=\left[\frac{\theta}{2}-\frac{\sin 2\theta}{4}\right]_0^{\frac{\pi}{6}}$	A1	AO3	(both correct)
	$\frac{\pi}{12}-\frac{\sin\frac{\pi}{3}}{4}-0+0=\frac{\pi}{12}-\frac{\sqrt{3}}{8}$	A1 **[14]**	AO3	

Question Number	Solution	Mark	AO	Notes
9.	$x^2 + 4 = 12 - x^2$ $2x^2 = 8$ $x = \pm 2$	M1 A1	AO3 AO3	(Equating y's)
	Area $= \int_{-2}^{2} \{12 - x^2 - (x^2 + 4)\}\ dx$	M1	AO3	(expressing area)
	$= \int_{-2}^{2} (8 - 2x^2)dx$			
	$= \left[8x - \frac{2x^3}{3}\right]_{-2}^{2}$	A2	AO3 AO3	(F.T arithmetic error)
	$= \frac{64}{3}$	A1	AO3	(c.a.o)
	Alternative mark scheme for the Area:			
	Area $= \int_{-2}^{2} (12 - x^2)dx - \int_{-2}^{2} (x^2 + 4)\,dx$	(M1)	(AO3)	
	$= \left[12x - \frac{x^3}{3} - \frac{x^3}{3} - 4x\right]_{-2}^{2}$	(A2)	(AO3) (AO3)	(A2 for 4 terms correct, A1 for 2 terms correct)
	$= \frac{64}{3}$	(A1) **[6]**	(AO3)	(c.a.o)

Question Number	Solution	Mark	AO	Notes
10. (a)	$f(x)=1+5x-x^4$ $f(1)=5,\ f(2)=-5$	M1	AO2	(Use of Intermediate Value Theorem.)
	There is a change of sign indicating there is a root between 1 and 2.	A1	AO2	(correct values and conclusions)
(b)	$x_{n+1}=\sqrt[4]{1+5x_n},\quad x_0=1\cdot5,\quad x_1=1.707476485$	B1	AO1	
	$x_2=1.75734609$	B1	AO1	
	$x_3=1.7687213,\quad x_4=1.7712854$ $x_5=1.771861948,\ \alpha\approx1.77$	B1	AO1	
(c)	$x_{n+1}=x_n-\dfrac{f(x_n)}{f'(x_n)}=x_n-\dfrac{1+5x_n-x_n^4}{5-4x_n^3}$	M1	AO1	Attempt to use Newton-Raphson
		A1	AO1	All terms correct
	$x_0=1.5$ $x_1=1.904411765$	M1 A1	AO1 AO1	
	$x_2=1.788115338$ $x_3=1.772305156$ $x_4=1.772029085$ $x_5=1.772028972$	A1	AO1	
	Root $\alpha\approx1.772029$	A1	AO1	Correct to 6 decimal places
		[11]		

Question Number	Solution	Mark	AO	Notes
11. (a)	$4x^3 + 2xy + x^2 \frac{dy}{dx} + 2y\frac{dy}{dx} = 0$	B2	AO1,AO1	(B2, 4 correct terms) (B1, 3 correct terms)
	Now, $x = -1, y = 3$ so that $-4 - 6 + \frac{dy}{dx} + 6\frac{dy}{dx} = 0$	B1	AO1	
	$\frac{dy}{dx} = \frac{10}{7}$	B1	AO1	
(b)	$\frac{dy}{dx} = \frac{dy}{dp} / \frac{dx}{dp} = \frac{2}{2p} = \frac{1}{p}$	M1 A1	AO1 AO1	
	Gradient of normal is $-p$	B1	AO1	
	Equation of normal is $(y - 2p) = -p(x - p^2)$	m1	AO1	
	$y - 2p = -px + p^3$			
	so that $y + px = 2p + p^3$	A1	AO1	convincing
	When $y = 0, \quad x = b$ $b = 2 + p^2$	B1	AO2	
	Since $p^2 > 0, \; b > 2$	E1	AO2	
		[11]		

Question Number	Solution	Mark	AO	Notes
12. (a)	Let $y = \cos x$			
	$\frac{dy}{dx} = \lim_{h \to 0}\left[\frac{\cos(x+h) - \cos x}{h}\right]$	M1	AO2	
	$= \lim_{h \to 0}\left[\frac{\cos x \cos h - \sin x \sin h - \cos x}{h}\right]$	A1	AO2	
	As h approaches 0 $\cos h \approx 1 - \frac{h^2}{2}$ and $\sin h \approx h$			
	So $\frac{dy}{dx} = \lim_{h \to 0}\left[\frac{\cos x\left(1 - \frac{h^2}{2}\right) - \sin x \times h - \cos x}{h}\right]$	M1	AO2	
	$= \lim_{h \to 0}\left[\frac{-\frac{h^2}{2}\cos x - h \sin x}{h}\right]$	A1	AO2	
	$= -\sin x$	A1	AO2	
(b) (i)	$\frac{(x^3+1)6x - 3x^2(3x^2)}{(x^3+1)^2}$	M1	AO1	(Correct formula)
	$= \frac{3x(2-x^3)}{(x^3+1)^2}$	A1	AO1	
(ii)	$3x^2 \tan 3x + 3x^3 \sec^2 3x$	M1	AO1	(Correct formula)
	$= 3x^2(\tan 3x + x \sec^2 3x)$	A1	AO1	(All Correct)
		[9]		

Question Number	Solution	Mark	AO	Notes
13. (a)	$\operatorname{cosec}^2 x + \cot^2 x = 5$			
	$1 + 2\cot^2 x = 5$	M1	AO1	(Attempt to write in terms of one function)
	$\cot^2 x = 2$	A1	AO1	
	$\tan x = \pm\frac{1}{\sqrt{2}}$	A1	AO1	
	$x = 35.3,\ 215.3^\circ, 144.7^\circ,\ 324.7^\circ$	B1,B1	AO1 AO1	(each pair)
(b) (i)	$4\sin\theta + 3\cos\theta \equiv R(\sin\theta\cos\alpha + \cos\theta\sin\alpha)$			
	$R\cos\alpha = 4$ $R\sin\alpha = 3$	B1 B1	AO1 AO1	
	$R = \sqrt{3^2 + 4^2} = 5$	B1	AO1	
	$\tan\alpha = \frac{3}{4},\quad \alpha = 36.87^\circ$	B1	AO1	
	$4\sin\theta + 3\cos\theta \equiv 5\sin(\theta + 36.87^\circ)$			
(ii)	$5\sin(\theta + 36.87^\circ) = 2$			
	$\sin(\theta + 36.87^\circ) = 0.4$	B1	AO1	
	$\theta + 36.87^\circ = 23.58^\circ, 156.42^\circ, 383.58^\circ$			
	$\theta = 119.5(5)^\circ,\ 346.7(1)^\circ$ $= 120^\circ, 347^\circ$ to the nearest degree	B1 B1	AO1 AO1	
		[12]		

Question Number	Solution	Mark	AO	Notes
14. (a)	$\frac{dV}{dt} = 4\frac{dh}{dt}$			
	$4\frac{dh}{dt} = 0.004 - 0.0008h$	M1	AO3	(3 terms, at least 2 correct)
	$\frac{dh}{dt} = 0.001 - 0.0002h$			
	$5000\frac{dh}{dt} = 5 - h$	A1	AO3	(Correct)
(b)	$5000\int \frac{dh}{5-h} = \int dt$	M1	AO1	(Separation of variables)
	$-5000\ \ln(5-h) = t + C$ (1)	A1,A1	AO1 AO1	(-1 if C omitted)
	$h = 0$ at $t = 0$			
	$\therefore -5000\ln(5) = C$	m1	AO1	
	Substitute in (1)			
	$-5000\ \ln(5-h) = t - 5000\ln(5)$			
	$t = 5000\ln\left(\frac{5}{5-h}\right)$	A1	AO1	
	$\therefore \left(\frac{5}{5-h}\right) = e^{\frac{t}{5000}}$	M1	AO1	(Attempt to invert)
	$5 - h = 5e^{\frac{-t}{5000}}$			
	$h = 5 - 5e^{\frac{-t}{5000}}$	A1	AO1	
(c)	$h = 5 - 5e^{\frac{-3600}{5000}}$			
	$= 2.57\text{ m}$	B1	AO1	
		[10]		

Question Number	Solution	Mark	AO	Notes
15.	$4x^2+9<12x$ $4x^2-12x+9<0$ $(2x-3)^2<0$ Impossible when x is real. Contradiction so that assumption is false. $\therefore 4x+\frac{9}{x}\geq 12$	M1 A1 A1 **[3]**	AO2 AO2 AO2	(Clear fractions)